Univerzitet u Zenici
Ekonomski fakultet
Odsjek: Menadžment preduzeća, Računovodstveni i revizijski menadžment
Akademska 2013/2014.

Sadržaj sveske sa vježbi iz
<u>**Matematike**</u>

(II dio sveske - sadrži gradivo od 9 do 15 sedmice)

<u>**Literatura za dodatno istraživanje:**</u>

• Matematika I za ekonomiste; Zečić, Huskanović, Alajbegović
• Zbirka zadataka iz više Matematike; Uščumlić, Miličić
• Zadaci i riješeni primjeri iz više matematike s primjenom na tehničke nauke; Demidović
• Zbirka zadataka iz Matematike; Stojanović
• Zbirka zadataka iz Matematičke analize; Berman

Dio tablice izvoda

1) $(c)'=0$; 2) $(u+v-w)'=u'+v'-w'$;
3) $(uv)'=u'v+v'u$; 3a) $(cu)'=cu'$;
4) $\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$; 4a) $\left(\frac{u}{c}\right)'=\frac{u'}{c}$;
4б) $\left(\frac{c}{v}\right)'=-\frac{cv'}{v^2}$; 5) $(x^n)'=nx^{n-1}$;
6) $(\sin x)'=\cos x$; 7) $(\cos x)'=-\sin x$;
8) $(\operatorname{tg} x)'=\sec^2 x$; 9) $(\operatorname{ctg} x)'=-\operatorname{cosec}^2 x$.

5) $(u^n)'=nu^{n-1}\cdot u'$; 8) $(\operatorname{tg} u)'=\sec^2 u\cdot u'$;
6) $(\sin u)'=\cos u\cdot u'$; 9) $(\operatorname{ctg} u)'=-\operatorname{cosec}^2 u\cdot u'$.
7) $(\cos u)'=-\sin u\cdot u'$;

10) $(a^u)'=a^u\ln a\cdot u'$; 11) $(\log u)'=\frac{u'}{u}\log e$;
10a) $(e^u)'=e^u u'$; 11a) $(\ln u)'=\frac{u'}{u}$;
10б) $(a^x)'=a^x\ln a$; 11б) $(\log x)'=\frac{1}{x}\log e$;
10в) $(e^x)'=e^x$; 11в) $(\ln x)'=\frac{1}{x}$.

12) $(\arcsin u)'=\frac{u'}{\sqrt{1-u^2}}$; 12a) $(\arcsin x)'=\frac{1}{\sqrt{1-x^2}}$;
13) $(\arccos u)'=-\frac{u'}{\sqrt{1-u^2}}$; 13a) $(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}$;
14) $(\operatorname{arc\,tg} u)'=\frac{u'}{1+u^2}$; 14a) $(\operatorname{arc\,tg} x)'=\frac{1}{1+x^2}$;
15) $(\operatorname{arc\,ctg} u)'=-\frac{u'}{1+u^2}$; 15a) $(\operatorname{arc\,ctg} x)'=-\frac{1}{1+x^2}$.

Dio tablice integrala

1. $\int u^a\,du=\frac{u^{a+1}}{a+1}+C,\ a\neq-1.$
2. $\int u^{-1}\,du=\int\frac{du}{u}=\int\frac{u'}{u}\,dx=$ $=\ln|u|+C.$
3. $\int a^u\,du=\frac{a^u}{\ln a}+C;\ \int e^u\,du=$ $=e^u+C.$
4. $\int \sin u\,du=-\cos u+C.$
5. $\int \cos u\,du=\sin u+C.$
6. $\int \sec^2 u\,du=\operatorname{tg} u+C.$
7. $\int \operatorname{cosec}^2 u\,du=-\operatorname{ctg} u+C.$
8. $\int\frac{du}{u^2+a^2}=\frac{1}{a}\operatorname{arc\,tg}\frac{u}{a}+C.$
9. $\int\frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$
10. $\int\frac{du}{\sqrt{a^2-u^2}}=\arcsin\frac{u}{a}+C.$
11. $\int\frac{du}{\sqrt{u^2+a}}=$ $=\ln\left|u+\sqrt{u^2+a}\right|+C.$

Funkcija dvije nezavisne promjenjive

Neka je S neprazan podskup prostora $\mathbb{R}^2$ i $T \subseteq \mathbb{R}$. Ako svakoj tački $M(x,y) \in S$ možemo unaprijed po datom pravilu f pridružiti jednu i samo jednu realnu vrijednost $z \in T$, tada kažemo da je data realna f-ja dvije realne promjenjive f iz $\mathbb{R}^2$ u $\mathbb{R}$ (sa skupa $S \subseteq \mathbb{R}$ u skup $T \subseteq \mathbb{R}$) i pišemo $z = f(x,y)$. Skup S na kojem je određena f-ja f naziva se domen ili definiciono područje f-je f (označavat ćemo ga sa $\mathcal{D}(f)$), a skup $f(A)$ skup vrijednosti f-je f ili kodomen (označavat ćemo ga sa $\mathcal{R}(f)$. Ako za f-ju, zadanu analitički (formulom) nije data oblast njene definisanosti, onda se pod njom podrazumjeva skup svih tačaka $M \in \mathbb{R}^2$ u kojoj f-ja, odnosno njen analitički izraz imaju određenu realnu vrijednost.

\# Za svaku od sljedećih f-ja, izračunati $f(3,2)$, i odrediti i skicirati domen.

a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$ b) $f(x,y) = x \ln(y^2 - x)$

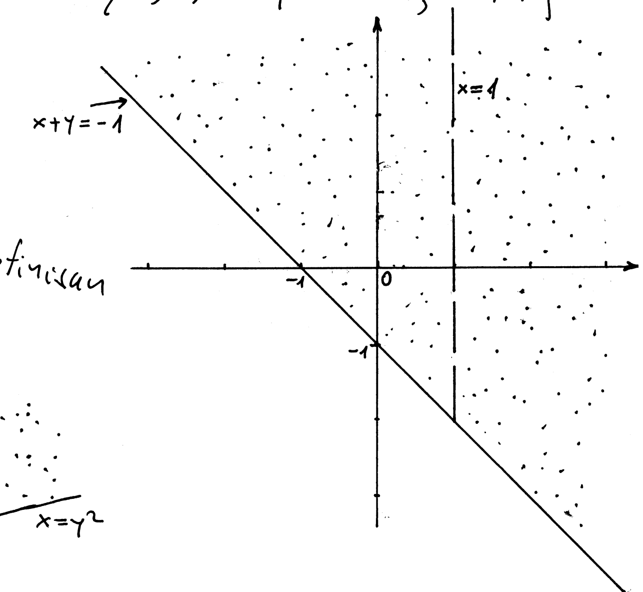
Rj.

a) $f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$

Izraz za f-ju $f(x,y)$ ima smisla ako je nazivnik različit od nule i ako je vrijednost pod korijenom nenegativna:

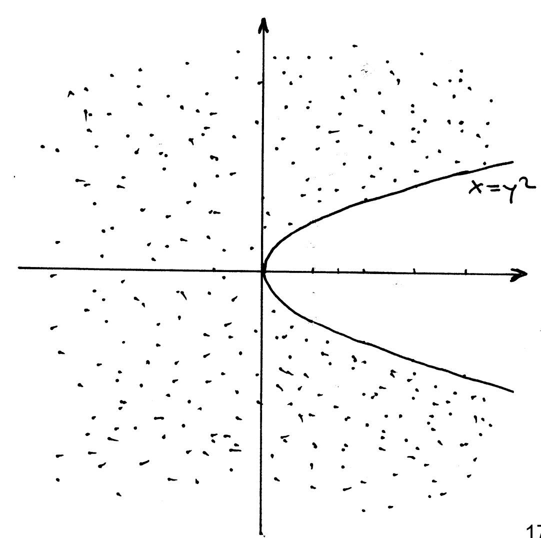
$$\begin{matrix} x-1 \neq 0 \\ x+y+1 \geq 0 \end{matrix} \Rightarrow \begin{matrix} x \neq 1 \\ x+y \geq -1 \end{matrix}$$

Domen f-je f je $D = \{(x,y) \in \mathbb{R}^2 \mid x+y \geq -1,\ x \neq 1\}$



b) $f(3,2) = 3\ln(2^2 - 3)$
$= 3\ln(4-3) = 3\underbrace{\ln 1}_{=0} =$
$= 0$

Izraz $\ln(y^2 - x)$ je definisan samo ako je $y^2 - x > 0$



$D = \{(x,y) \mid x < y^2\}$

Odrediti domen i rang f-je $g(x,y)=\sqrt{9-x^2-y^2}$

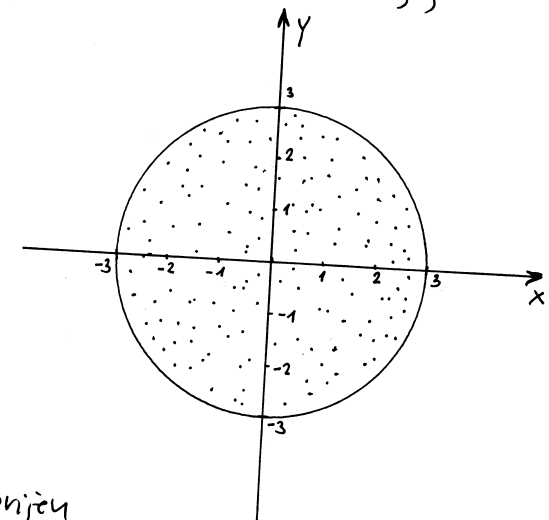
Rj.

F-ja ima smisla akko $9-x^2-y^2 \geq 0$

$$x^2+y^2 \leq 9$$

Domen f-je $g(x,y)$ je $D=\{(x,y)\in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}$

(znamo da je $x^2+y^2=9$ krug sa centrom u tački $C(0,0)$ poluprečnika $r=3$).



Rang f-je g je

$\{z\in \mathbb{R} \mid z=\sqrt{9-x^2-y^2},\ (x,y)\in D\}$

Primjetimo da je

$$9-x^2-y^2 \leq 9 \quad \text{za } \forall (x,y)\in D$$

pa je $\sqrt{9-x^2-y^2} \leq 3$

z je pozitivan kvadratni korijen

$$z \geq 0$$

Prema tome, rang f-je $g(x,y)$ je

$$\{z \mid 0 \leq z \leq 3\} = [0,3]$$

Skicirati graf f-je $f(x,y)=6-3x-2y$.

Rj.

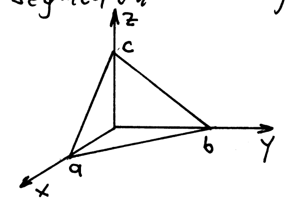
Graf f-je $f(x,y)$ ima jednačinu $z=6-3x-2y$

$$3x+2y+z=6$$

ovo predstavlja ravan.

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

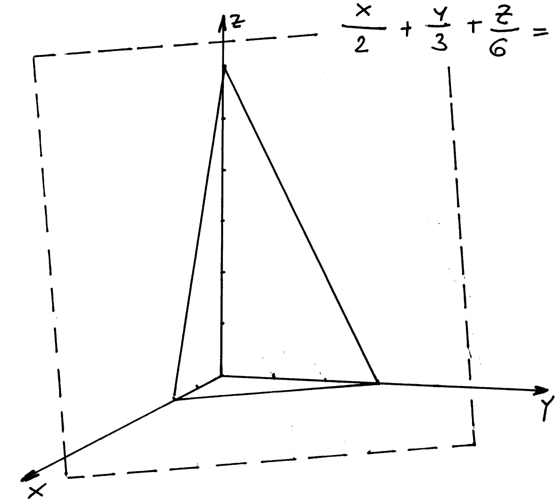
segmentni oblik jednačine ravni



U našem slučaju

$3x+2y+z=6 \quad /:6$

$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$



Skicirati graf f-je $g(x,y)=\sqrt{9-x^2-y^2}$.

Rj. Graf f-je ima jednačinu $z=\sqrt{9-x^2-y^2}$

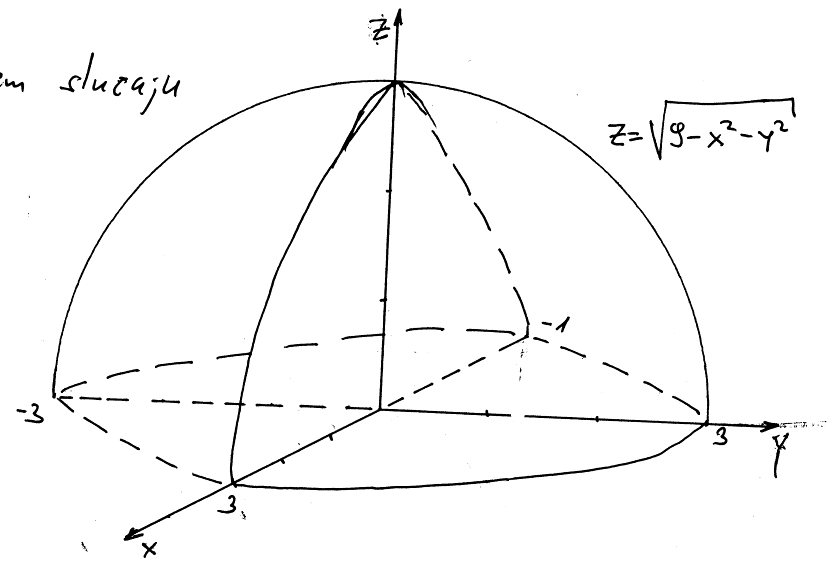
$$z=\sqrt{9-x^2-y^2} \quad /^2$$

$$z^2=9-x^2-y^2$$

$$x^2+y^2+z^2=9$$

$x^2+y^2+z^2=R^2$
je jednačina sfere sa centrom u koordinatnom početku poluprečnika R

U našem slučaju



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Ekstremne vrijednosti f-ja dviju promjenjivih

Neka je data f-ja $z = f(x,y)$.

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0$$

SISTEM

rješenjem sistema dobijemo stacionarne tačke koje mogu ali i ne moraju biti ekstrem

npr. $M(p_1, p_2)$ je jedna stacionarna tačka.

$A = \frac{\partial^2 z(p_1,p_2)}{\partial x^2}$

$B = \frac{\partial^2 z(p_1,p_2)}{\partial x \partial y}$

$C = \frac{\partial^2 z(p_1,p_2)}{\partial y^2}$

$D = AC - B^2$

$D > 0$ f-ja ima ekstrem u tački $M(p_1,p_2)$

a) $A > 0$ imamo z_{min}

b) $A < 0$ imamo z_{max}

$D < 0$ f-ja nema ekstrem

$D = 0$ potrebno ispitati ponašanje f-je u okolini stacionarne tačke:

$\Delta z(M) = z(p_1 + \varepsilon, p_2 + \omega) - z(p_1, p_2)$ –priraštaj f-je

$\Delta z \geq 0 \quad \forall \varepsilon ; \forall \omega \Rightarrow$ u tački M f-ja ima minimum

$\Delta z \leq 0 \quad \forall \varepsilon ; \forall \omega \Rightarrow$ u tački M f-ja ima maksimum

\# Naći ekstreme f-je $z = x^2 - 2x - y - \ln(2-y) + 4$.

$\mathcal{D}: 2 - y > 0$

R:

$\frac{\partial z}{\partial x} = 2x - 2$

$\frac{\partial z}{\partial y} = -1 - \frac{1}{2-y} \cdot (-1) = \frac{1}{2-y} - 1$

$2x - 2 = 0$

$\frac{1}{2-y} - 1 = 0$

$x = 1, \quad y = 1$

Tačka $M(1,1)$ je stacionarna tačka (kandidat za ekstrem)

$(2-y)^{-1}$

$\frac{\partial^2 z}{\partial x^2} = 2$

$\frac{\partial^2 z}{\partial x \partial y} = 0$

$\frac{\partial^2 z}{\partial y^2} = (-1)(2-y)^{-2} \cdot (-1) = \frac{1}{(2-y)^2}$

$M(1,1)$

$A = 2, \; B = 0, \; C = 1$

$D = AC - B^2 = 2 > 0$

F-ja ima ekstrem.

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}(1,1) = 1 - 2 - 1 - \ln 1 + 4 = -2 + 4 = 2$

Nađi ekstreme f-je $z = x^3 - 5xy + 5y^2 + 7x - 15y$.

Rj. Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 3x^2 - 5y + 7 \qquad \frac{\partial z}{\partial y} = 10y - 5x - 15$$

$$\begin{aligned} 3x^2 - 5y + 7 &= 0 \quad /\cdot 2 \\ -5x + 10y - 15 &= 0 \\ \hline 6x^2 - 10y + 14 &= 0 \\ -5x + 10y - 15 &= 0 \quad + \\ \hline \end{aligned}$$

$6x^2 - 5x - 1 = 0$

$D = 25 + 24 = 49$

$x_{1,2} = \frac{5 \pm 7}{2 \cdot 6}$

$x_1 = \frac{-2}{2 \cdot 6} = \frac{-1}{6}, \quad x_2 = \frac{12}{12} = 1$

$6(x + \frac{1}{6})(x - 1) = 0$

Za $x_1 = -\frac{1}{6} \Rightarrow -5 \cdot (-\frac{1}{6}) + 10y - 15 = 0$

$10y = 15 - \frac{5}{6}$

$10y = \frac{90 - 5}{6} = \frac{85}{6}$

$y = \frac{\cancel{85}^{17}}{\cancel{10}_{2} 6} = \frac{17}{12}$

$x_2 = 1 \Rightarrow -5 + 10y - 15 = 0$

$10y = 20$

$y = 2$

Stacionarne tačke su $(1,2)$ i $(-\frac{1}{6}, \frac{17}{12})$

$$\frac{\partial^2 z}{\partial x^2} = 6x \qquad \frac{\partial^2 z}{\partial x \partial y} = -5 \qquad \frac{\partial^2 z}{\partial y^2} = 10$$

Za $M_1(1,2)$

$A = 6, B = -5, C = 10, \; D = AC - B^2 = 60 - 25 > 0$

f-ja ima ekstrem

$A > 0 \Rightarrow$ fja ima minimum

$z_{min}(1,2) = 1 - 10 + 20 + 7 - 30 = 8 + 10 - 30 = 8 - 20 = -12$

Za $M_2(-\frac{1}{6}, \frac{17}{12})$

$A = -1, \; B = -5, \; C = 10, \; D = AC - B^2 = -10 - 25 = -35$

Fja u ovoj tački nema ekstrem

Nađi ekstreme f-je $z = x + y - \frac{3}{2} \ln(x^2 + y^2 + 1)$.

Rj. Izračunajmo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{\cancel{2}} \cdot \frac{1}{x^2 + y^2 + 1} \cdot \cancel{2}x = 1 - \frac{3x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{\cancel{2}} \cdot \frac{1}{x^2 + y^2 + 1} \cdot \cancel{2}y = 1 - \frac{3y}{x^2 + y^2 + 1}$$

Nađimo stacionarne tačke

$$\begin{aligned} 1 - \frac{3x}{x^2 + y^2 + 1} &= 0 \\ 1 - \frac{3y}{x^2 + y^2 + 1} &= 0 \\ \hline 3x = 3y &\Rightarrow x = y \end{aligned}$$

$1 - \frac{3x}{2x^2 + 1} = 0 \quad / \cdot 2x^2 + 1$

$2x^2 + 1 - 3x = 0$

$2x^2 - 3x + 1 = 0$

$D = 9 - 8 = 1$

$x_{1,2} = \frac{3 \pm 1}{4}$

$x_1 = \frac{2}{4} = \frac{1}{2}$

$x_2 = 1$

Stacionarne tačke su $M_1(\frac{1}{2}, \frac{1}{2})$ i $M_2(1,1)$

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = 0 - 3 \cdot \frac{1 \cdot (x^2 + y^2 + 1) - x \cdot 2x}{(x^2 + y^2 + 1)^2} = -3 \cdot \frac{-x^2 + y^2 + 1}{(x^2 + y^2 + 1)^2} = 3 \frac{x^2 - y^2 - 1}{(x^2 + y^2 + 1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0 - 3x \cdot (-1)(x^2 + y^2 + 1)^{-2} \cdot 2y = 6 \frac{xy}{(x^2 + y^2 + 1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 0 - 3 \frac{1 \cdot (x^2 + y^2 + 1) - y \cdot 2y}{(x^2 + y^2 + 1)^2} = -3 \frac{x^2 - y^2 + 1}{(x^2 + y^2 + 1)^2}$$

Za $M_1(\frac{1}{2}, \frac{1}{2})$, $A = 3 \cdot \frac{-1}{(\frac{1}{2} + 1)^2} = \frac{-3}{\frac{9}{4}} = \frac{-12}{9} = -\frac{4}{3}$, $B = \frac{2}{3}$, $C = -\frac{4}{3}$

$D = AC - B^2 = \frac{16}{9} - \frac{4}{9} > 0$ f-ja ima ekstrem u tački M_1

$A < 0$ f-ja ima minimum $z_{min}(\frac{1}{2}, \frac{1}{2}) = 1 - \frac{3}{2} \ln \frac{3}{2}$

Za $M_2(1,1)$, $A = -\frac{1}{3}$, $B = \frac{2}{3}$, $C = -\frac{1}{3}$

$D = AC - B^2 = \frac{1}{9} - \frac{4}{9} < 0$ fja u tački M_2 nema ekstrem

Naći ekstreme f-je $z = x + y - \frac{3}{2}\ln(x^2+y^2+1)$.

Rj: Pronađimo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{2}\cdot\frac{2x}{x^2+y^2+1} = 1 - \frac{3x}{x^2+y^2+1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{2}\cdot\frac{2y}{x^2+y^2+1} = 1 - \frac{3y}{x^2+y^2+1}$$

Pronađimo stacionarne tačke

$$\left.\begin{array}{l} \frac{\partial z}{\partial x} = 0 \Rightarrow 1 = \frac{3x}{x^2+y^2+1} \\ \frac{\partial z}{\partial y} = 0 \Rightarrow 1 = \frac{3y}{x^2+y^2+1} \end{array}\right\} \Rightarrow x = y \text{ (deljenjem jednačina)}$$

Sad imamo $x = y$ i $1 = \frac{3x}{x^2+y^2+1} \Rightarrow 1 = \frac{3x}{2x^2+1} \Rightarrow 2x^2 - 3x + 1 = 0$

$D = 9 - 8 = 1$

$x_1 = 1,\ x_2 = \frac{1}{2}$

Stacionarne tačke su $M_1(1,1)$ i $M_2(\frac{1}{2}, \frac{1}{2})$.

Pronađimo druge parcijalne izvode.

$$\frac{\partial^2 z}{\partial x^2} = \left(1 - \frac{3x}{x^2+y^2+1}\right)'_x = \frac{-3(x^2+y^2+1) + 3x\cdot 2x}{(x^2+y^2+1)^2} = \frac{3x^2 - 3y^2 - 3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(1 - \frac{3y}{x^2+y^2+1}\right)'_y = \left|\begin{array}{l}\text{zbog}\\ \text{simetričnosti}\end{array}\right| = \frac{3y^2 - 3x^2 - 3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{3x\cdot 2y}{(x^2+y^2+1)^2} = \frac{6xy}{(x^2+y^2+1)^2}$$

Za tačku $M_1(1,1)$: $A = -\frac{3}{9} = -\frac{1}{3}$, $B = \frac{6}{9} = \frac{2}{3}$, $C = -\frac{3}{9} = -\frac{1}{3}$, $D = AC - B^2$

$D = \frac{1}{9} - \frac{4}{9} < 0 \Rightarrow$ u M_1 f-ja nema ekstrema

Za tačku $M_2(\frac{1}{2}, \frac{1}{2})$: $A = \frac{-3}{(\frac{3}{2})^2} = -\frac{3}{\frac{9}{4}} = -\frac{12}{9} = -\frac{4}{3} \Rightarrow C = -\frac{4}{3}$

$B = \frac{\frac{3}{2}}{\frac{9}{4}} = \frac{12}{18} = \frac{2}{3}$, $D = AC - B^2 = \frac{16}{9} - \frac{4}{9} = \frac{12}{9} = \frac{4}{3} > 0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A < 0 \Rightarrow$ u M_2 f-ja ima maximum. $z_{max}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} + \frac{1}{2} - 3\ln(\frac{1}{4} + \frac{1}{4} + 1) = 1 - \ln\frac{27}{8}$

Naći ekstreme f-je $z = \frac{8}{x} + \frac{x^2}{y} + y + 1$.

Rj: Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 8\cdot(-1)x^{-2} + 2\frac{x}{y} = \frac{-8}{x^2} + 2\frac{x}{y}$$

$$\frac{\partial z}{\partial y} = x^2\cdot(-1)y^{-2} + 1 = \frac{-x^2}{y^2} + 1$$

$-\frac{8}{x^2} + \frac{2x}{y} = 0$

$-\frac{x^2}{y^2} + 1 = 0$

$\frac{8}{x^2} - 2\frac{x}{y} = 0$

$\frac{x^2}{y^2} = 1 \Rightarrow \left(\frac{x}{y}\right)^2 = 1$

Prema tome $\frac{x}{y} = 1$ i $\frac{x}{y} = -1$

Za $\frac{x}{y} = 1 \Rightarrow \frac{8}{x^2} - 2\cdot 1 = 0$

$\frac{8}{x^2} = 2 \quad /\cdot x^2\ (x \neq 0)$

$2x^2 = 8$

$x^2 = 4$

$x_1 = -2,\ x_2 = 2$

$x_1 = -2 \Rightarrow \frac{x}{y} = 1$

$y = -2$

$(-2, -2)$

Za $x_2 = 2 \Rightarrow$

$\Rightarrow \frac{x}{y} = 1$

$y_2 = 2$

$(2, 2)$

Za $\frac{x}{y} = -1$ imamo

$\frac{8}{x^2} + 2 = 0$

$\frac{8}{x^2} = -2 \quad /\cdot x^2\ (x \neq 0)$

$-2x^2 = 8$

ova jednačina nema rešenja u skupu realnih brojeva

Stacionarne tačke su $M_1(-2,-2)$ i $M_2(2,2)$.

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = (-8)(-2)x^{-3} + \frac{2}{y} = \frac{16}{x^3} + \frac{2}{y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\cdot(-1)y^{-2} = \frac{-2x}{y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2\cdot(-2)y^{-3} = \frac{2x^2}{y^3}$$

Za $M_1(-2,-2)$

$A = \frac{16}{-8} + \frac{2}{-2} = -2 - 1 = -3$

$B = \frac{-2\cdot(-2)}{4} = 1$, $C = \frac{2\cdot 4}{-8} = -1$

$D = AC - B^2 = 3 - 1 = 2 > 0$.

f-ja u tački $M_1(-2,-2)$ ima ekstrem,

$A < 0$ f-ja ima maksimum

$z_{max}(-2,-2) = -4 - 2 - 2 + 1 = -7$

Za $M_2(2,2)$

$A = 2 + 1 = 3$, $B = \frac{-4}{4} = -1$, $C = \frac{8}{8} = 1$

$D = AC - B^2 = 3 - 1 = 2 > 0$ f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}(2,2) = 4 + 2 + 2 + 1 = 9$

Naći ekstreme f-je $z=(x^2+y)\sqrt{e^y}$.

Rj. $\frac{\partial z}{\partial x} = 2x\sqrt{e^y}$

$$\frac{\partial z}{\partial y} = \sqrt{e^y} + (x^2+y)\frac{1}{2\sqrt{e^y}}\cdot e^y = \sqrt{e^y} + (x^2+y)\cdot\frac{1}{2}\sqrt{e^y}$$

$$= \left(\frac{1}{2}x^2 + \frac{1}{2}y + 1\right)\sqrt{e^y} = \frac{1}{2}(x^2+y+2)\sqrt{e^y}$$

$$\begin{aligned} 2x\sqrt{e^y} &= 0 \\ \frac{1}{2}(x^2+y^2+1)\sqrt{e^y} &= 0 \end{aligned}$$

$e^y>0\ \forall y\in R$

prema tome $x=0$

$\sqrt{e^y}>0\ \forall x\in R$

$x^2+y+2=0$

$x=0 \Rightarrow y+2=0$

$y=-2$

$M(0,-2)$ je stacionarna tačka (kandidat za ekstrem)

$$\frac{\partial^2 z}{\partial x^2} = 2\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial x\partial y} = 2x\frac{1}{2\sqrt{e^y}}\cdot e^y = x\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{2}\sqrt{e^y} + \left(\frac{1}{2}x^2+\frac{1}{2}y+1\right)\frac{e^y\cdot\sqrt{e^y}}{2\sqrt{e^y}\cdot\sqrt{e^y}} = \frac{1}{2}\sqrt{e^y}\left(\frac{1}{2}x^2+\frac{1}{2}y+2\right)$$

$M(0,-2)$

$A = 2\sqrt{e^{2}} = 2\frac{1}{\sqrt{e^2}}$

$B=0$

$C=\frac{1}{2}\sqrt{e^{-2}}\left(\frac{1}{2}0+\underbrace{\frac{1}{2}(-2)}_{-1}+2\right) = \frac{1}{2}\sqrt{\frac{1}{e^2}}$

$D=AC-B^2=\frac{2}{\sqrt{e^2}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{e^2}}=\frac{1}{e^2}$

$D>0 \Rightarrow$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$$z_{min}(0,-2)=(0-2)\sqrt{e^{-2}}=(-2)\cdot\frac{1}{\sqrt{e^2}}\approx -0.7358$$

Naći ekstreme f-je $z=e^{-2x^2}(x-y^2)$.

Rj. Nađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = e^{-2x^2}\cdot(-4)x(x-y^2)+e^{-2x^2}\cdot 1 = e^{-2x^2}(-4x^2+4xy^2+1)$$

$$\frac{\partial z}{\partial y} = e^{-2x^2}\cdot(-2)y = -2ye^{-2x^2}$$

$$\begin{aligned} e^{-2x^2}(-4x^2+4xy^2+1) &= 0 \\ -2y\,e^{-2x^2} &= 0 \end{aligned}$$

e^{-2x^2} je uvijek pozitivno

$$\begin{aligned} -4x^2+4xy^2+1&=0 \\ -2y&=0 \Rightarrow y=0 \end{aligned}$$

$-4x^2+1=0$

$x^2=\frac{1}{4} \Rightarrow x_1=-\frac{1}{2},\ x_2=\frac{1}{2}$

Stacionarne tačke su $M_1(-\frac{1}{2},0)$ i $M_2(\frac{1}{2},0)$

$$\frac{\partial^2 z}{\partial x^2} = e^{-2x^2}\cdot(-4x)(-4x^2+4xy^2+1)+e^{-2x^2}(-8x+4y^2)=$$

$$=e^{-2x^2}(16x^3-16x^2y^2-4x-8x+4y^2)=e^{-2x^2}(16x^3-16x^2y^2-12x+4y^2)$$

$$=4e^{-2x^2}(4x^3-4x^2y^2-3x+y^2)$$

$$\frac{\partial^2 z}{\partial x\partial y}=e^{-2x^2}(8xy)=8xye^{-2x^2}$$

$$\frac{\partial^2 z}{\partial y^2}=-2e^{-2x^2}$$

Za tačku $M_1(-\frac{1}{2},0)$

$A=4e^{-2\cdot\frac{1}{4}}\left(4\cdot\left(-\frac{1}{8}\right)-4\cdot\frac{1}{4}\cdot 0-3\cdot\left(-\frac{1}{2}\right)+0\right)=4e^{-\frac{1}{2}}\left(-\frac{1}{2}+\frac{3}{2}\right)=\frac{4}{\sqrt{e}}$

$B=0,\ C=-2e^{-2\cdot\frac{1}{4}}=-2e^{-\frac{1}{2}}=\frac{-2}{\sqrt{e}}$

$D=AC-B^2=\frac{-8}{e}<0$

f-ja z u tački M_1 nema ekstrem

Za tačku $M_2(\frac{1}{2},0)$

$A=4e^{-2\cdot\frac{1}{4}}\left(4\cdot\frac{1}{8}-0-3\cdot\frac{1}{2}+0\right)=$
$=4e^{-\frac{1}{2}}\left(\frac{1}{2}-\frac{3}{2}\right)=\frac{-4}{\sqrt{e}}$

$B=0,\ C=-2e^{-2\cdot\frac{1}{4}}=\frac{-2}{\sqrt{e}}$

$D=AC-B^2=\frac{8}{e}>0 \Rightarrow$ f-ja z u tački M_2 ima ekstrem

$A<0 \Rightarrow z_{max}(\frac{1}{2},0)=e^{-2\cdot\frac{1}{4}}\left(\frac{1}{2}-0\right)=\frac{1}{2}\cdot e^{-\frac{1}{2}}=\frac{1}{2\sqrt{e}}$

\# Odrediti ekstremne vrijednosti f-je

$$z=\frac{xy}{2}+(47-x-y)\left(\frac{x}{3}+\frac{y}{4}\right).$$

Rj.

$$\frac{\partial z}{\partial x}=\frac{1}{2}y+(-1)\left(\frac{x}{3}+\frac{y}{4}\right)+(47-x-y)\cdot\frac{1}{3}=\frac{1}{2}y-\frac{1}{3}x-\frac{1}{4}y+\frac{47}{3}-\frac{1}{3}x-\frac{1}{3}y$$

$$=-\frac{2}{3}x+\frac{6-3-4}{12}y+\frac{47}{3}=-\frac{2}{3}x-\frac{1}{12}y+\frac{47}{3}$$

$$\frac{\partial z}{\partial y}=\frac{1}{2}x+(-1)\left(\frac{x}{3}+\frac{y}{4}\right)+(47-x-y)\cdot\frac{1}{4}=\frac{1}{2}x-\frac{1}{3}x-\frac{1}{4}y+\frac{47}{4}-\frac{1}{4}x-\frac{1}{4}y$$

$$=-\frac{1}{12}x-\frac{1}{2}y+\frac{47}{4}$$

$$-\frac{2}{3}x-\frac{1}{12}y+\frac{47}{3}=0 \quad /\cdot 12$$
$$-\frac{1}{12}x-\frac{1}{2}y+\frac{47}{4}=0 \quad /\cdot 12$$

$$-8x-y+188=0$$
$$-x-6y+141=0$$

$$-8x-y+188=0$$
$$x=-6y+141$$

$$-8(-6y+141)-y+188=0$$
$$48y-1128-y+188=0$$
$$47y=940$$
$$y=20$$
$$x=-6y+141=-120+141=21$$

Stacionarna tačka je $M(21,20)$.

$$\frac{\partial^2 z}{\partial x^2}=-\frac{2}{3}\qquad \frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{12}\qquad \frac{\partial^2 z}{\partial y^2}=-\frac{1}{2}$$

$D=AC-B^2$

$M(21,20)$

$A=-\frac{2}{3},\ B=-\frac{1}{12},\ C=-\frac{1}{2}$

$$D=\frac{2}{6}-\frac{1}{144}=\frac{1}{3}-\frac{1}{144}>0$$

f-ja z ima ekstrem

$A<0$ f-ja ima maksimum

$$z_{max}(21,20)=21\cdot 10+(47-41)(7+5)=210+6\cdot 12=210+72=282$$

$z_{max}(21,20)=282$ traženi ekstrem f-je

Nađi ekstreme f-je $z=x^4+y^4-2x^2$.

Rj.

$$\frac{\partial z}{\partial x}=4x^3-4x \qquad \frac{\partial z}{\partial y}=4y^3$$

$$4x^3-4x=0 \quad /:4$$
$$4y^3=0 \quad /:4$$
$$x^3-x=0$$
$$y^3=0$$

$$x(x^2-1)=0$$
$$y^3=0$$
$$x(x-1)(x+1)=0$$
$$y^3=0$$

$y=0 \ \wedge\ (x_1=0,\ x_2=1,\ x_3=-1)$

Stacionarne tačke f-je su $M_1(-1,0)$, $M_2(0,0)$ i $M_3(1,0)$.

$$\frac{\partial^2 z}{\partial x^2}=12x^2-4 \qquad \frac{\partial^2 z}{\partial x\partial y}=0 \qquad \frac{\partial^2 z}{\partial y^2}=12y^2$$

za $M_1(-1,0)$, $A=8$, $B=0$, $C=0$

$D=0$ ispitujemo ponašanje f-je u okolini tačke $M_1(-1,0)$

$$\Delta z=z(-1+\varepsilon,0+\omega)-z(-1,0)=$$
$$=(-1+\varepsilon)^4+\omega^4-2(-1+\varepsilon)^2-\left[(-1)^4+0^4-2(-1)^2\right]$$
$$=1-4\varepsilon+6\varepsilon^2-4\varepsilon^3+\varepsilon^4+\omega^4-2(1-2\varepsilon+\varepsilon^2)-(1-2)$$
$$=1-4\varepsilon+6\varepsilon^2-4\varepsilon^3+\varepsilon^4+\omega^4-2+4\varepsilon-2\varepsilon^2+1$$
$$=\varepsilon^4-4\varepsilon^3+4\varepsilon^2+\omega^4=\varepsilon^2(\varepsilon^2-4\varepsilon+4)+\omega^4$$
$$=\varepsilon^2(\varepsilon-2)^2+\omega^4\geq 0 \text{ za } \forall\varepsilon \text{ i } \forall\omega$$

paskalov trougao

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

f-ja ima minimum u tački $M_1(-1,0)$, $z_{min}=-1$

za $M_2(0,0)$, $A=-4$, $B=0$, $C=0$, $D=AC-B^2=0$

ispitujemo ponašanje f-je u okolini tačke

$$\Delta z=z(0+\varepsilon,0+\omega)-z(0,0)=\varepsilon^4+\omega^4-2\varepsilon^2=\varepsilon^2(\varepsilon^2-2)+\omega^4$$

$\varepsilon=0:\quad \Delta z=\omega^4$

$\omega=0:\quad \Delta z=\varepsilon^2(\varepsilon^2-2)\ \Rightarrow\ \Delta z<0$ za $\varepsilon^2<2$; $\Delta z>0$ za $\varepsilon^2>2$

Priraštaj f-je je promjenjivog znaka pa f-ja nema ekstrem u tački M_2

za $M_3(1,0)$, $A=8$, $B=0$, $C=0$, $D=AC-B^2=0$ ispitujemo ponašanje f-je u okolini tačke

$$\Delta z=z(1+\varepsilon,0+\omega)-z(1,0)=(1+\varepsilon)^4+\omega^4-2(1+\varepsilon)^2-(1-2)$$
$$=1+4\varepsilon+6\varepsilon^2+4\varepsilon^3+\varepsilon^4+\omega^4-2-4\varepsilon-2\varepsilon^2+1=\varepsilon^4+4\varepsilon^3+4\varepsilon^2+\omega^4$$

$\Delta z=\varepsilon^2(\varepsilon+2)^2+\omega^4\geq 0\ \ \forall\varepsilon$ i $\forall\omega$ f-ja z u tački M_3 ima min $z_{min}=-1$

Naći ekstreme f-je $z=(2x^2+3y^2)e^{-(x^2+y^2)}$.

Rj.

$$\frac{\partial z}{\partial x} = 4x\cdot e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2x) = (4x-4x^3-6xy^2)e^{-x^2-y^2}$$

$$\frac{\partial z}{\partial y} = 6y e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2y) = (6y-4x^2y-6y^3)e^{-x^2-y^2}$$

$$2x(2-2x^2-3y^2)e^{-x^2-y^2}=0$$
$$2y(3-2x^2-3y^2)e^{-x^2-y^2}=0$$

$e^{-x^2-y^2}\neq 0 \quad \forall(x,y\in\mathbb{R})$

$x=0$ i $y=0$, $M_1(0,0)$

ili

$x=0$ i $3-2x^2-3y^2=0$

$3y^2=3$, $y^2=1$, $y_{1,2}=\pm1$

$M_2(0,-1)$, $M_3(0,1)$

ili $2-2x^2-3y^2=0$ i $y=0$

$2x^2=2$, $x^2=1$, $x_{1,2}=\pm1$

$M_4(-1,0)$, $M_5(1,0)$

ili

$$\begin{aligned} &2-2x^2-3y^2=0\\ -\ &3-2x^2-3y^2=0\\ \hline &-1=0\end{aligned}$$

sistem nema rješenja

Stacionarne tačke su M_1, M_2, M_3, M_4 i M_5.

$$\frac{\partial^2 z}{\partial x^2} = (4-12x^2-6y^2)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2x) = (8x^4+12x^2y^2-20x^2-6y^2+4)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = (-12xy)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2y) = (-20xy+8x^3y+12xy^3)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = (6-4x^2-18y^2)e^{-x^2-y^2}+(6y-4x^2y-6y^3)e^{-x^2-y^2}\cdot(-2y) = (-30y^2+12y^4+8x^2y^2-4x^2+6)e^{-x^2-y^2}$$

za $M_1(0,0)$, $A=4$, $B=0$, $C=6$, $D=AC-B^2=24>0$ ima ekstrem

$A>0$ ima minimum, $z_{min}(0,0)=0$

za $M_2(0,-1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum, $z_{max}(0,-1)=3e^{-1}$

za $M_3(0,1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum $z_{max}(0,1)=3e^{-1}$

za $M_4(-1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$, $D=AC-B^2=-16e^{-2}<0$

f-ja u tački $M_4(-1,0)$ nema ekstrem

za $M_5(1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$

f-ja u tački $M_5(1,0)$ nema ekstrem

Naći stacionarne tačke f-je $z=xy\ln(x^2+y^2)$.

Rj.

$$\frac{\partial z}{\partial x} = y\ln(x^2+y^2)+xy\cdot\frac{1}{x^2+y^2}\cdot 2x = y\ln(x^2+y^2)+\frac{2x^2y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = x\ln(x^2+y^2)+xy\cdot\frac{1}{x^2+y^2}\cdot 2y = x\ln(x^2+y^2)+\frac{2xy^2}{x^2+y^2}$$

$$y\ln(x^2+y^2)+\frac{2x^2y}{x^2+y^2}=0$$
$$x\ln(x^2+y^2)+\frac{2xy^2}{x^2+y^2}=0$$

$$y\left(\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}\right)=0$$
$$x\left(\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}\right)=0$$

$y=0$ ili $\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}=0$

$x=0$ ili $\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=0$

$y=0$ i $x=0$ $M_1(0,0)$ za M_1 f-ja nije definisana

ili

$y=0$ i $\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=0$

$\ln x^2=0$

$x^2=1$

$x_1=-1$, $x_2=1$

$M_2(-1,0)$, $M_3(1,0)$

ili

$\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}=0$ i $x=0$

$\ln y^2=0$ $M_4(0,-1)$

$y_1=-1$ $y_2=1$ $M_5(0,1)$

ili

$$\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}=0 \quad (1)$$
$$\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=0 \quad (2)$$

(1)-(2): $\frac{2x^2}{x^2+y^2}-\frac{2y^2}{x^2+y^2}=0$

$2x^2-2y^2=0$

$2(x^2-y^2)=0$

$2(x-y)(x+y)=0$

$x=y$ ili $x=-y$.

za $x=y$: $\ln(2x^2)+1=0$

$\ln(2x^2)=-1$

$e^{-1}=2x^2$

$x^2=\frac{1}{2e}$

$x_{1,2}=\pm\frac{1}{\sqrt{2e}}$

$M_6\left(-\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$, $M_7\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$

za $y=-x$: $\ln(2x^2)+1=0$

$x_{1,2}=\pm\frac{1}{\sqrt{2e}}$

$M_8\left(-\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$

$M_9\left(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$

Stacionarne tačke su: $M_2, M_3, M_4, M_5, M_6, M_7, M_8$ i M_9.

Uslovni ekstremi f-ja dviju promjenjivih

Ako trebamo naći ekstrem f-je $z=f(x,y)$ tako da x i y zadovoljavaju neki uslov $g(x,y)=0$ tada tražimo ekstrem Lagranžove f-je $F(x,y,\lambda)=f(x,y)+\lambda g(x,y)$.

$$\left.\begin{aligned} \frac{\partial F}{\partial x}&=0\\ \frac{\partial F}{\partial y}&=0\\ \frac{\partial F}{\partial z}&=0 \end{aligned}\right\} \text{SISTEM}$$

rješavanjem sistema dobićemo neke stacionarne tačke i dalji proces se nastavlja kao kod traženja ekstrema f-je dvije promjenjive

II način: neka je $M(p_1,p_2)$ neka stacionarna tačka

$$d^2F(p_1,p_2)=F''_{xx}(p_1,p_2)\,dx^2+2F''_{xy}(p_1,p_2)\,dx\,dy+F''_{yy}(p_1,p_2)\,dy^2$$

$$d^2F(p_1,p_2)>0 \quad\Rightarrow\quad z_{min}(p_1,p_2)$$

$$d^2F(p_1,p_2)<0 \quad\Rightarrow\quad z_{max}(p_1,p_2)$$

Ako se desi slučaj da imamo više uslova, onda uvodimo više parametara $(\lambda,\mu,\dots)$.

1. Naći ekstreme f-je $z=6-4x-3y$ uz uslov $x^2+y^2=1$.

Rj. $F(x,y)=6-4x-3y+\lambda(x^2+y^2-1)\cancel{=0}$

$$\frac{\partial F}{\partial x}=-4+2\lambda x \qquad \frac{\partial F}{\partial y}=-3+2\lambda y \qquad \frac{\partial F}{\partial \lambda}=x^2+y^2-1$$

$$\begin{aligned} 2\lambda x-4&=0\\ 2\lambda y-3&=0\\ x^2+y^2-1&=0 \end{aligned} \qquad \begin{aligned} 2\lambda x&=4\\ 2\lambda y&=3\\ x^2+y^2&=1 \end{aligned}$$

$$x=\frac{2}{\lambda},\quad y=\frac{3}{2\lambda},\quad x^2+y^2=1$$

$$\frac{4}{\lambda^2}+\frac{9}{4\lambda^2}=1 \qquad \frac{25}{4\lambda^2}=1 \qquad 4\lambda^2=25 \qquad \lambda_{1,2}=\pm\frac{5}{2}$$

$$\lambda_1=-\frac{5}{2} \Rightarrow x_1=-\frac{4}{5};\quad y_1=\frac{3}{2\cdot\left(-\frac{5}{2}\right)}=-\frac{3}{5}$$

$$\lambda_2=\frac{5}{2} \Rightarrow x_2=\frac{2}{\frac{5}{2}}=\frac{4}{5};\quad y_2=\frac{3}{2\cdot\frac{5}{2}}=\frac{3}{5}$$

Stacionarne tačke su $M\left(-\frac{4}{5},-\frac{3}{5}\right)$ za $\lambda=-\frac{5}{2}$ i $N\left(\frac{4}{5},\frac{3}{5}\right)$ za $\lambda=\frac{5}{2}$.

$$\frac{\partial^2F}{\partial x^2}=2\lambda \qquad \frac{\partial^2F}{\partial x\partial y}=0 \qquad \frac{\partial^2F}{\partial y^2}=2\lambda$$

za $M\left(-\frac{4}{5},-\frac{3}{5}\right)$, $\lambda=-\frac{5}{2}$

$A=-5$, $B=0$, $C=-5$, $D=AC-B^2=25>0$

f-ja ima ekstrem, $A<0$ f-ja ima maksimum

$$z_{max}\left(-\frac{4}{5},-\frac{3}{5}\right)=6-4\left(-\frac{4}{5}\right)-3\left(-\frac{3}{5}\right)=\frac{30+16+9}{5}=\frac{55}{5}=11$$

za $N\left(\frac{4}{5},\frac{3}{5}\right)$, $\lambda=\frac{5}{2}$, $A=5$, $B=0$, $C=5$, $D=AC-B^2=25>0$

f-ja ima ekstrem u tački N, $A>0$ f-ja ima minimum

$$z_{min}\left(\frac{4}{5},\frac{3}{5}\right)=6-4\cdot\frac{4}{5}-3\cdot\frac{3}{5}=\frac{30-16-9}{5}=\frac{5}{5}=1$$

2. Naći uslovne ekstreme f-je $z=y+2x+3$ uz uslov $x^2-6x+y+5=0$.

Rj. $F(x,y)=2x+y+3+\lambda(x^2-6x+y+5)$

$$\frac{\partial F}{\partial x}=2+2\lambda x-6\lambda \qquad \frac{\partial F}{\partial y}=1+\lambda \qquad \frac{\partial F}{\partial \lambda}=x^2-6x+y+5$$

$$\begin{aligned} 2\lambda x-6\lambda+2&=0 \quad /:2\\ \lambda+1&=0\\ x^2-6x+y+5&=0 \end{aligned} \qquad \begin{aligned} \lambda x&=3\lambda-1\\ \lambda&=-1\\ x^2-6x+y+5&=0 \end{aligned}$$

$$-x=-3-1 \qquad x=4 \qquad x^2-6x+y+5=0 \qquad 16-24+y+5=0 \qquad y=3$$

Tačka $M(4,3)$ je stacionarna tačka, za $\lambda=-1$

$$\frac{\partial^2F}{\partial x^2}=2\lambda \qquad \frac{\partial^2F}{\partial x\partial y}=0 \qquad \frac{\partial^2F}{\partial y^2}=0$$

$M(4,3)$, $\lambda=-1$

$A=-2$, $B=0$, $C=0$ $\Rightarrow$ $D=AC-B^2=0$

$$d^2F=\frac{\partial^2F}{\partial x^2}dx^2+\frac{\partial^2F}{\partial x\partial y}\cancel{dx\,dy}+\frac{\partial^2F}{\partial y^2}dy^2$$

$$d^2F=2\lambda\,dx^2 \quad\Rightarrow\quad d^2F=-2\,dx^2<0$$

U tački $M(4,3)$ f-ja ima maksimum, $z_{max}(4,3)=3+8+3=14$

3. Odrediti ekstreme f-je $z=x^2+y^2$ uz uslov $\frac{x}{2}+\frac{y}{3}=1$.

Rj. $z_{min}\left(\frac{18}{13},\frac{12}{13}\right)=\frac{36}{13}$, $\lambda=-\frac{72}{13}$

4. Naći uslovne ekstreme f-je $z=\ln(x+y)$, ako je $x^2+2y^2=4$.

Rj. $z_{max}\left(2\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)=\ln\left(3\sqrt{\frac{2}{3}}\right)$, $\lambda=-\frac{1}{8}$.

Naći uslovne ekstreme f-je $z=2x^4+8y^4+24$ ako je $8x+4y=1$.

Rj. $F(x,y,\lambda)=2x^4+8y^4+24+\lambda(8x+4y-1)$

$$\frac{\partial F}{\partial x}=8x^3+8\lambda \qquad \frac{\partial F}{\partial y}=32y^3+4\lambda \qquad \frac{\partial F}{\partial \lambda}=8x+4y-1$$

$8x^3+8\lambda=0 \quad /:8$

$32y^3+4\lambda=0 \quad /:4$

$x^3+\lambda=0$

$-\ 8y^3+\lambda=0$

$x^3-8y^2=0$

$x^3=8y^3$

$x=2y$

$8x+4y-1=0$

$8\cdot 2y+4y-1=0$

$20y=1$

$y=\frac{1}{20}$

$x=2\cdot\frac{1}{20}=\frac{1}{10}$

$M_1\left(\frac{1}{10},\frac{1}{20}\right)$ je stacionarna tačka

$$\frac{\partial^2 F}{\partial x^2}=24x^2 \qquad \frac{\partial^2 F}{\partial x\partial y}=0 \qquad \frac{\partial^2 F}{\partial y^2}=96y^2$$

$D=AC-B^2$

$M_1\left(\frac{1}{10},\frac{1}{20}\right)$

$A=24\cdot\frac{1}{100}=\frac{24}{100}=\frac{12}{50}=\frac{6}{25}$

$B=0$

$C=\cancel{96}\cdot\frac{1}{20\cdot 20}=\frac{24}{100}=\frac{12}{50}=\frac{6}{25}$

$D=\left(\frac{6}{25}\right)^2>0$ f-ja ima ekstrem

$A>0$ f-ja ima minimum

$$Z_{min}\left(\frac{1}{10},\frac{1}{20}\right)=2\cdot\frac{1}{10^4}+8\cdot\frac{1}{20^4}+24=\frac{2}{10\,000}+\frac{\cancel{8}}{20\cdot 20\cdot 20\cdot 20}+24$$

$$=\frac{2}{10\,000}+\frac{1}{20\,000}+\frac{24\,000}{1\,000}=\frac{4+1+480\,000}{20\,000}=$$

$$=\frac{480\,005}{20\,000}\ \frac{:5}{:5}=\frac{96\,001}{4\,000}$$

$Z_{min}=\frac{96\,001}{4\,000}$ je minimum f-je u tački $M\left(\frac{1}{10},\frac{1}{20}\right)$

Naći uslovne ekstreme f-je $z=(x-y)^4+1$ ako je $x^2+y^2=18$.

Rj. $F(x,y,\lambda)=(x-y)^4+1+\lambda(x^2+y^2-18)$

$$\frac{\partial F}{\partial x}=4(x-y)^3+2\lambda x \qquad \frac{\partial F}{\partial y}=4(x-y)^3\cdot(-1)+2\lambda y \qquad \frac{\partial F}{\partial \lambda}=x^2+y^2-18$$

$4(x-y)^3+2\lambda x=0 \quad \ldots(1)$

$-4(x-y)^3+2\lambda y=0 \quad \ldots(2)$

$x^2+y^2-18=0 \quad \ldots(3)$

(1) (2): $2\lambda x+2\lambda y=0 \quad /:2$

$\lambda(x+y)=0$

$\lambda=0$ ili $x+y=0$

a) $x+y=0$

$x=-y$

$(-y)^2+y^2=18$

$2y^2=18$

$y_{1,2}=\pm 3$

$y_1=-3 \Rightarrow x_1=3$

$y_2=3 \Rightarrow x_2=-3$

$M_1(3,-3)$, $M_2(-3,3)$

za M_1 (1) $\Rightarrow 6\lambda=-4\cdot 6^3$

za M_2 (1) $\Rightarrow -6\lambda=-4(-6)^3$

$\Rightarrow \lambda=-144$

b) $\lambda=0$

(1) $\Rightarrow 4(x-y)^3=0$

$x=y$

$2y^2=18$

$y_{3,4}=\pm 3$

$M_3(-3,-3)$

$M_4(3,3)$

$\lambda=0$

Stacionarne tačke su $M_1(3,-3)$ $M_2(-3,3)$ za $\lambda=-144$ i $M_3(-3,-3)$ i $M_4(3,3)$ za $\lambda=0$.

$$\frac{\partial^2 F}{\partial x^2}=12(x-y)^2+2\lambda \qquad \frac{\partial^2 F}{\partial x\partial y}=-12(x-y)^2 \qquad \frac{\partial^2 F}{\partial y^2}=12(x-y)^2+2\lambda$$

$D=AC-B^2$

$M_1(3,-3)$, $\lambda=-144$

$A=12\cdot 36-2\cdot 144=144$

$B=-12\cdot 36=-432$

$C=144$

$D=20\,736-186\,624$

f-ja u tački M_1 nema ekstrema

$M_2(-3,3)$, $\lambda=-144$

$\left.\begin{array}{l}A=144\\ B=432\\ C=144\end{array}\right\}\ D<0$

f-ja u tački M_2 nema ekstrem

$M_3(-3,-3)$, $\lambda=0$

$A=0$, $B=0$, $C=0$ $\Rightarrow$ $D=0$ potrebno je ispitati f-ju u okolini tačke $M_3(-3,-3)$

$$\Delta z(M_3)=z(-3+\varepsilon,-3+\omega)-z(-3,-3)=(-3+\varepsilon+3-\omega)^4+1-1=(\varepsilon-\omega)^4>0\quad \forall\varepsilon \text{ i } \forall\omega$$

Priraštaj f-je u okolini tačke M_3 je pozitivan pa f-ja u M_3 ima minimum, $z_{min}(-3,-3)=1$

$M_4(3,3)$, $\lambda=0$

$A=0$, $B=0$, $C=0$ $\Rightarrow$ $D=0$ potrebno je ispitati f-ju u okolini tačke $M_4(3,3)$

$$\Delta z(M_4)=z(3+\varepsilon,3+\omega)-z(3,3)=(3+\varepsilon-3-\omega)^4+1-1=(\varepsilon-\omega)^4>0\quad \forall\varepsilon \text{ i } \forall\omega$$

Priraštaj f-je u okolini tačke M_4 je pozitivan pa f-ja u M_4 ima minimum, $z_{min}(3,3)=1$.

(#) Naći uslovne ekstreme f-je $z=2x+4y$ ako je $\frac{2}{x}+\frac{4}{y}=3$.

Rj: Formirajmo Lagranžovu f-ju $F(x,y,\lambda)=2x+4y+\lambda\left(\frac{2}{x}+\frac{4}{y}-3\right)$.

$$\frac{\partial F}{\partial x}=2+2\lambda\cdot\frac{(-1)}{x^2} \qquad \left(\frac{1}{x}\right)'=(x^{-1})'=(-1)(x^{-2}) \qquad (x^{-2})'=(-2)x^{-3}=\frac{-2}{x^3}$$

$$\frac{\partial F}{\partial y}=4+4\lambda\cdot\frac{(-1)}{y^2}$$

$$\frac{\partial F}{\partial \lambda}=\frac{2}{x}+\frac{4}{y}-3$$

Formirajmo sistem

$$4-\frac{4\lambda}{y^2}=0 \quad /:4$$
$$2-\frac{2\lambda}{x^2}=0 \quad /:2$$
$$\frac{2}{x}+\frac{4}{y}=3$$

$$1-\frac{\lambda}{x^2}=0 \qquad 1=\frac{\lambda}{x^2} \quad (1)$$
$$1-\frac{\lambda}{y^2}=0 \qquad 1=\frac{\lambda}{y^2} \quad (2)$$
$$\frac{2}{x}+\frac{4}{y}=3 \qquad \frac{2}{x}+\frac{4}{y}=3 \quad (3)$$

(1) i (2) $\Rightarrow \frac{\lambda}{x^2}=\frac{\lambda}{y^2} \Rightarrow x^2=y^2$

tj. $x=\pm y$

za $x=y$ iz (3) $\frac{2}{x}+\frac{4}{x}=3$

$\frac{6}{x}=3 \Rightarrow x=2 \Rightarrow y=2$

za $x=-y$ iz (3)

$\frac{2}{x}-\frac{4}{x}=3 \Rightarrow -\frac{2}{x}=3$

$3x=-2 \Rightarrow x=-\frac{2}{3}$

$\Rightarrow y=\frac{2}{3}$

za $M_1(2,2) \Rightarrow 2-2\lambda\cdot\frac{1}{4}=0$

$\lambda=4$

za $M_2\left(-\frac{2}{3},\frac{2}{3}\right) \Rightarrow 2-2\lambda\cdot\frac{9}{4}=0 \Rightarrow \lambda=\frac{4}{9}$

Stacionarne tačke su $M_1(2,2)$ za $\lambda=4$ i $M_2\left(-\frac{2}{3},\frac{2}{3}\right)$ za $\lambda=\frac{4}{9}$.

$$\frac{\partial^2 F}{\partial x^2}=\frac{4\lambda}{x^3}, \qquad \frac{\partial^2 F}{\partial x\partial y}=0, \qquad \frac{\partial^2 F}{\partial y^2}=\frac{8\lambda}{y^3}$$

Za $M_1(2,2)$, $\lambda=4$

$A=\frac{16}{8}=2$, $B=0$, $C=\frac{32}{8}=4$, $D=AC-B^2=8>0$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$z_{min}(2,2)=4+8=12$

Za $M_2\left(-\frac{2}{3},\frac{2}{3}\right)$, $\lambda=\frac{4}{9}$, $A=\frac{\frac{16}{9}}{-\frac{8}{27}}=-\frac{16\cdot 27}{8\cdot 9}=-2\cdot 3=-6$

$B=0$, $C=\frac{\frac{32}{9}}{\frac{8}{27}}=\frac{32\cdot 27}{8\cdot 9}=4\cdot 3=12$, $D=AC-B^2=-72<0 \Rightarrow$

$\Rightarrow$ f-ja u tački M_2 nema ekstremnu vrijednost

(#) Naći uslovne ekstreme f-je $z=xy$ ako je $x^2+y^2=2ax$, $a>0$.

Rj: Posmatramo f-ju $F(x,y,\lambda)=xy+\lambda(x^2+y^2-2ax)$

$$\frac{\partial F}{\partial x}=y+2\lambda x-2a\lambda$$
$$\frac{\partial F}{\partial y}=x+2\lambda y$$
$$\frac{\partial F}{\partial \lambda}=x^2+y^2-2ax$$

$$y+2\lambda x-2a\lambda=0$$
$$x+2\lambda y=0$$
$$x^2+y^2-2a\cdot x=0$$

(1) $y+2\lambda(x-a)=0 \Rightarrow x-a=\frac{-y}{2\lambda}$ ...(*)

(2) $x=-2\lambda y$

(3) $x^2-2x\cdot a+a^2-a^2+y^2=0$

(2) u (1): $y+2\lambda(-2\lambda y-a)=0$

(3): $(x-a)^2+y^2=a^2$

$y-4\lambda^2 y-2a\lambda=0$

$y(1-4\lambda^2)=2a\lambda$

$y=\frac{2a\lambda}{1-4\lambda^2}$

(*) u (3):

$$\frac{y^2}{4\lambda^2}+y^2=a^2$$
$$y^2\left(\frac{1}{4\lambda^2}+1\right)=a^2$$
$$y^2\left(\frac{1+4\lambda^2}{4\lambda^2}\right)=a^2$$
$$y^2=\frac{4a^2\lambda^2}{1+4\lambda^2}$$
$$y=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}}$$

$$y=\frac{2a\lambda}{1-4\lambda^2}=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}} \Rightarrow 1-4\lambda^2=\pm\sqrt{1+4\lambda^2}$$

$$(1-4\lambda^2)^2=1+4\lambda^2$$
$$16\lambda^4-8\lambda^2+1=1+4\lambda^2$$
$$16\lambda^4-12\lambda^2=0$$
$$\lambda^2(16\lambda^2-12)=0$$

$\lambda_1=0$

$\lambda_{2,3}=\pm\sqrt{\frac{12}{16}}=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt{3}}{2}$

$\lambda_1=0$: $y=0$, $x=0$

$\lambda_2=\frac{\sqrt{3}}{2}$:

$$y+\sqrt{3}x-a\sqrt{3}=0$$
$$x+y\sqrt{3}=0 \quad /\cdot\sqrt{3}$$
$$x^2+y^2-2ax=0$$

$$\sqrt{3}x+y=a\sqrt{3}$$
$$-\ \sqrt{3}x+3y=0$$
$$-2y=a\sqrt{3} \qquad x=-\frac{3}{2}a$$
$$y=\frac{-a}{2}\sqrt{3}$$

$\lambda_3=-\frac{\sqrt{3}}{2}$:

$$y-x\sqrt{3}+a\sqrt{3}=0$$
$$x-y\sqrt{3}=0 \quad /\cdot\sqrt{3}$$
$$x^2+y^2-2ax=0$$

$$-x\sqrt{3}+y=-a\sqrt{3}$$
$$+\ x\sqrt{3}-3y=0$$
$$-2y=-a\sqrt{3}$$
$$y=\frac{\sqrt{3}}{2}a \Rightarrow x=\frac{3}{2}a$$

Stacionarne tačke su $M_1(0,0)$ za $\lambda=0$, $M_2(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a)$ za $\lambda=\frac{\sqrt{3}}{2}$; $M_3(\frac{3}{2}a, \frac{\sqrt{3}}{2}a)$ za $\lambda=-\frac{\sqrt{3}}{2}$.

$\frac{\partial^2 F}{\partial x^2}=2\lambda$

$\frac{\partial^2 F}{\partial x \partial y}=1$

$\frac{\partial^2 F}{\partial y^2}=2\lambda$

$M_1(0,0)$, $\lambda=0$

$D=AC-B^2=-1<0 \Rightarrow$ f-ja u tački $M_1(0,0)$ nema ekstrem

$M_2(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a)$, $\lambda=\frac{\sqrt{3}}{2}$

$D=AC-B^2=3-1=2>0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A=\sqrt{3}>0 \Rightarrow$ f-ja ima minimum

$z_{min}(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a)=-\frac{3\sqrt{3}}{4}a^2$

$M_3(\frac{3}{2}a, \frac{\sqrt{3}}{2}a)$ za $\lambda=-\frac{\sqrt{3}}{2}$

$D=AC-B^2=3-1>0 \Rightarrow$ f-ja ima ekstrem

$A=-\sqrt{3}<0 \Rightarrow$ f-ja u tački M_3 ima maksimum

$z_{max}(\frac{3}{2}a, \frac{\sqrt{3}}{2}a)=\frac{3\sqrt{3}}{4}a^2$

Dio tablice integrala

1. $\int u^a\,du=\frac{u^{a+1}}{a+1}+C,\ a\neq-1.$
2. $\int u^{-1}\,du=\int\frac{du}{u}=\int\frac{u'}{u}\,dx=$ $=\ln|u|+C.$
3. $\int a^u\,du=\frac{a^u}{\ln a}+C;\ \int e^u\,du=$ $=e^u+C.$
4. $\int \sin u\,du=-\cos u+C.$
5. $\int \cos u\,du=\sin u+C.$
6. $\int \sec^2 u\,du=\operatorname{tg} u+C.$
7. $\int \operatorname{cosec}^2 u\,du=-\operatorname{ctg} u+C.$
8. $\int\frac{du}{u^2+a^2}=\frac{1}{a}\operatorname{arc\,tg}\frac{u}{a}+C.$
9. $\int\frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$
10. $\int\frac{du}{\sqrt{a^2-u^2}}=\arcsin\frac{u}{a}+C.$
11. $\int\frac{du}{\sqrt{u^2+a}}=$ $=\ln\left|u+\sqrt{u^2+a}\right|+C.$

Neodređeni integral

1. Primitivna f-ja i neodređeni integral. Osnovne formule integriranja.

Određivanje f-je $F(x)$ iz datog diferencijala $dF(x)=f(x)\,dx$ (ili iz neke date derivacije $F'(x)=f(x)$) nazivamo integriranje, a traženu f-ju $F(x)$ nazivamo primitivna f-ja f-je $f(x)$. Drugim riječima efekat suprotan diferenciranju nazivamo integriranje.

Navedimo nekoliko primjera primitivnih f-ja:

- $F(x)=\cos x$ je primitivna f-ja f-je $f(x)=\sin x$ zato što je
  $F'(x)=f(x)$ $\quad ((\cos x)'=\sin x)$ ili
  $dF(x)=f(x)\,dx$ $\quad (d(\cos x)=\sin x\,dx)$
- $F(x)=\frac{1}{4}x^4$ je primitivna f-ja f-je $f(x)=x^3$ zato što je
  $F'(x)=f(x)$ $\quad \left(\left(\frac{1}{4}x^4\right)'=\frac{1}{4}\cdot 4x^3=x^3\right)$ ili
  $dF(x)=f(x)\,dx$ $\quad \left(d\left(\frac{1}{4}x^4\right)=\frac{1}{4}d(x^4)=\frac{1}{4}\cdot 4x^3dx=x^3dx\right)$
- $F(x)=\operatorname{tg}x$ je primitivna f-ja f-je $f(x)=\frac{1}{\cos^2x}$ zato što je
  $F'(x)=f(x)$ $\quad \left((\operatorname{tg}x)'=\frac{1}{\cos^2x}\right)$ ili
  $dF(x)=f(x)\,dx$ $\quad \left(d(\operatorname{tg}x)=\frac{1}{\cos^2x}dx\right)$.
- $F(x)=\arcsin x$ je primitivna f-ja f-je $f(x)=\frac{1}{\sqrt{1-x^2}}$ zato što je
  $dF(x)=f(x)\,dx$ $\quad \left(d(\arcsin x)=\frac{1}{\sqrt{1-x^2}}dx\right)$ ili
  $F'(x)=f(x)$ $\quad \left((\arcsin x)'=\frac{1}{\sqrt{1-x^2}}\right)$.
- $F(x)=\ln|x|$ je primitivna f-ja f-je $f(x)=\frac{1}{x}$.
  ZA VJEŽBU OBJASNITI ZAŠTO.

Svaka neprekidna f-ja f-je $f(x)$ ima beskonačno mnogo različitih primitivnih f-ja, koje se jedna od druge razlikuju u članu koji predstavlja konstantu: ako je $F(x)$ primitivna f-ja f-je $f(x)$ (tj. ako je $F'(x)=f(x)$) tada je i $F(x)+c$ primitivna f-ja od $f(x)$, gdje je c proizvoljna konstanta. Zašto? Zato što

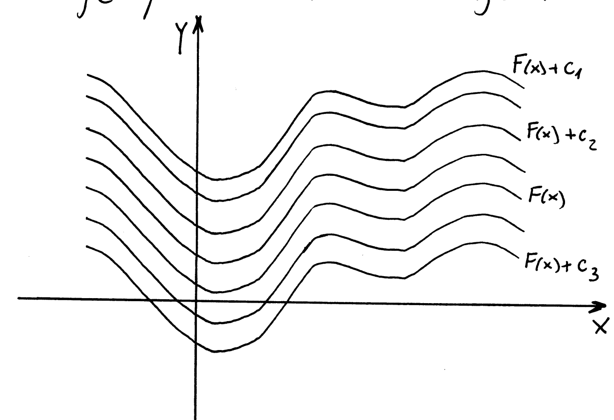
$$(F(x)+c)'=F'(x)=f(x).$$

Opšti izraz $F(x)+c$ skupa svih primitivnih f-ja f-je $f(x)$ zovemo neodređeni integral f-je $f(x)$ i označavamo ga sa znakom $\int$:

$$\int f(x)\,dx=F(x)+c \quad \text{akko} \quad d[F(x)+c]=f(x)\,dx$$

(ako i samo ako)

Geometrijski, u xOy koordinatnom sistemu, grafici svih primitivnih f-ja date f-je $f(x)$ predstavljaju familiju krivih, koje zavise od parametra c, i koje se mogu izvesti jedna iz druge paralelnom translacijom duž Y-ose.



Osobine neodređenog integrala:

I $\quad \frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$ ili $d\int f(x)\,dx = f(x)\,dx$

(izvod integrala) (diferencijal integrala)

II $\quad \int F'(x)\,dx = F(x)+c$ ili $\int dF(x) = F(x)+c$

III $\quad \int a\,f(x)\,dx = a\int f(x)\,dx$ tj. konstantu a koja množi f-ju možemo izvesti ispred znaka integrala

IV $\quad \int [f_1(x)+f_2(x)-f_3(x)]\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx - \int f_3(x)\,dx$

tj. integral sume je jednak sumi integrala svih članova

Osnovne formule integriranja:

1. $\int u^a\,du = \frac{u^{a+1}}{a+1}+c, \quad a\neq -1$
2. $\int u^{-1}\,du = \int \frac{du}{u} = \ln|u|+c$
3. $\int a^u\,du = \frac{a^u}{\ln a}+c$

$\int e^u\,du = e^u + c$

4. $\int \sin u\,du = -\cos u + c$
5. $\int \cos u\,du = \sin u + c$
6. $\int \frac{du}{\cos^2 u} = tg\,u + c$
7. $\int \frac{du}{\sin^2 u} = ctg\,u + c$
8. $\int \frac{du}{u^2+a^2} = \frac{1}{a}\,arctg\,\frac{u}{a} + c$
9. $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + c$
10. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + c$
11. $\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right| + c$

U osnovnim formulama integriranja a predstavlja konstantu, u je nezavisna promjenjiva ili bilo koja (diferencijabilna) f-ja neke nezavisne promjenjive. Navedimo nekoliko primjera korištenja osnovnih formula integriranja:

- Integral $I_1 = \int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx$ predstavlja formulu 1 $\left(\int u^a\,du = \frac{u^{a+1}}{a+1}+c, \ a\neq -1\right)$ gdje su $u=x$, $a=\frac{1}{2}$. Prema toj formuli $I_1 = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{x^3}+c$.
- Integral $I_2 = \int 3^x\,dx$ predstavlja formulu 3 $\left(\int a^u\,du = \frac{a^u}{\ln a}+c\right)$ gdje su $a=3$, $u=x$. Prema toj formuli $I_2 = \frac{3^x}{\ln 3}+c$.
- Integral $I_3 = \int \frac{dt}{t^2+3}$ predstavlja formulu 8 $\left(\int \frac{du}{u^2+a^2} = \frac{1}{a}\,arctg\,\frac{u}{a}+c\right)$ gdje su $u=t$, $a=\sqrt{3}$. Prema toj formuli $I_3 = \frac{1}{\sqrt{3}}\,arctg\,\frac{t}{\sqrt{3}}+c$.
- Integral $I_4 = \int \frac{d\varphi}{\sqrt{\varphi^2-5}}$ predstavlja formulu 11 $\left(\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right|+c\right)$ gdje su $u=\varphi$, $a=-5$. Prema toj formuli $I_4 = \ln\left|\varphi+\sqrt{\varphi^2-5}\right|+c$
- Integral $I_5 = \int \frac{2x}{x^2+7}\,dx = \int \frac{(x^2+7)'}{x^2+7}\,dx = \int \frac{d(x^2+7)}{x^2+7}$ predstavlja formulu 2 $\left(\int \frac{du}{u} = \ln|u|+c\right)$ pri čemu je $u=x^2+7$ (zato što je $d(x^2+7)=2x\,dx$). Prema toj formuli $I_5 = \ln(x^2+7)+c$. Znak apsolutne vrijednosti smo izostavili zato što je uvijek $x^2+7>0$.

U opštem slučaju, u formulama 2, 9 i 11

$\left(\int \frac{du}{u} = \ln|u| + C, \quad \int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| + C \quad i \quad \int \frac{du}{\sqrt{u^2+a}} = \ln|u+\sqrt{u^2+a}| + C\right)$

pišemo apsolutnu vrijednost samo u slučaju kada izraz ispod logaritma može imati negativnu vrijednost.

- Integral $I_6 = \int 5 \sin 5t \, dt = \int \sin 5t \, d(5t)$ predstavlja formulu 4 $\left(\int \sin u \, du = -\cos u + C\right)$ pri čemu je $u = 5t$. Prema toj formuli $I_6 = -\cos 5t + C$.

- Integral $I_7 = \int e^{\sin\varphi} \cos\varphi \, d\varphi = \int e^{\sin\varphi} d\sin\varphi$ (zato što $d\sin\varphi = \cos\varphi \, d\varphi$) predstavlja formulu 3 $\left(\int e^u du = e^u + C\right)$, pri čemu je $u = \sin\varphi$. Možemo zaključiti $I_7 = e^{\sin\varphi} + C$.

- Posmatrajmo integral $I_8 = \int \frac{e^x dx}{e^{2x}-1}$. Primjetimo da, zato što je $de^x = e^x dx$ možemo pisati $I_8 = \int \frac{e^x dx}{e^{2x}-1} = \int \frac{de^x}{(e^x)^2 - 1}$. Prema formuli 9 $\left(\int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln\left|\frac{u-a}{u-a}\right|\right)$ pri čemu je $u = e^x$, $a = 1$ možemo zaključiti $I_8 = \frac{1}{2} \ln \frac{|e^x - 1|}{e^x + 1} + C$.

(#) Naći slijedeće integrale i provjeriti rezultat diferenciranjem

a) $\int \frac{dx}{x^3}$ b) $\int \frac{dx}{\sqrt{2-x^2}}$ c) $\int 3^t 5^t dt$ d) $\int \sqrt{y+1} \, dy$

e) $\int \frac{dx}{2x^2 - 6}$

Rj: a) $\int \frac{dx}{x^3} = \int x^{-3} dx = \frac{x^{-2}}{-2} + C = C - \frac{1}{2x^2}$

Koristili smo formulu $\int u^\alpha du = \frac{u^{\alpha+1}}{\alpha+1} + C$ gdje je $u = x$, $\alpha = -3$.

Provjera:

Diferencirajmo dobijenu f-ju

$$d\left(C - \frac{1}{2x^2}\right) = -\frac{1}{2}(x^{-2})' dx = \left(-\frac{1}{2}\right)(-2)\, x^{-3} dx = x^{-3} dx$$
$$= \frac{dx}{x^3}$$

b) $\int \frac{dx}{\sqrt{2-x^2}} = \arcsin \frac{x}{\sqrt{2}} + C$

Koristili smo formulu $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$ gdje je $u = x$, $a = \sqrt{2}$

Provjera:

$d\left(\arcsin \frac{x}{\sqrt{2}} + C\right) =$

$$=\left(\arcsin\frac{x}{\sqrt{2}}\right)' dx = \frac{\left(\frac{x}{\sqrt{2}}\right)'}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\,dx = \frac{dx}{\sqrt{2-x^2}}.$$

c) $$\int 3^t 5^t dt = \int 15^t = \frac{15^t}{\ln 15} + C$$

Koristili smo formulu $\int a^u du = \frac{a^u}{\ln a} + C$ pri čemu je $a=15$, $u=t$

Provjera: $$d\left(\frac{15^t}{\ln 15}+C\right) = \frac{1}{\ln 15}\left(15^t\right)' dt = \frac{1}{\ln 15} 15^t \ln 15\, dt$$

$$= 15^t dt$$

d) $$\int \sqrt{y+1}\, dy = \int (y+1)^{\frac{1}{2}} d(y+1) = \frac{(y+1)^{\frac{3}{2}}}{\frac{3}{2}} + C =$$

$$= \frac{2}{3}\sqrt{(y+1)^3} + C$$

Koristili smo formulu $\int u^\alpha du = \frac{u^{\alpha+1}}{\alpha+1} + C$ pri čemu je $u=y+1$, $\alpha = \frac{1}{2}$ $(d(y+1)=dy)$

Provjera:

$$d\left(\frac{2}{3}\sqrt{(y+1)^3}+C\right) = \frac{2}{3}\left(\sqrt{(y+1)^3}\right)' dy = \frac{2}{3}\cdot\frac{3}{2}(y+1)^{\frac{3}{2}-1} dy$$

$$= \sqrt{y+1}\, dy$$

e) $$\int \frac{dx}{2x^2-6} = \frac{1}{2}\int\frac{dx}{x^2-3} = \frac{1}{2}\cdot\frac{1}{2\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C$$

Koristili smo formulu $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$ pri čemu je $u=x$, $a=\sqrt{3}$

Provjera:

$$d\left(\frac{1}{4\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|+C\right) = \frac{1}{4\sqrt{3}}\left(\ln\frac{x-\sqrt{3}}{x+\sqrt{3}}\right)' dx$$

$$= \frac{1}{4\sqrt{3}}\left(\ln(x-\sqrt{3}) - \ln(x+\sqrt{3})\right)' dx =$$

$$= \frac{1}{4\sqrt{3}}\cdot\left(\frac{1}{x-\sqrt{3}} - \frac{1}{x+\sqrt{3}}\right) dx = \frac{1}{4\sqrt{3}}\cdot\frac{x+\sqrt{3}-x+\sqrt{3}}{x^2-3}\,dx$$

$$= \frac{dx}{2(x^2-3)}$$

(#) Odrediti integrale

a) $\int \frac{dx}{\sqrt[3]{5x}}$ b) $\int \frac{dt}{\sqrt{3-4t^2}}$ c) $\int \cos 3\varphi \, d\varphi$

d) $\int e^{-\frac{x}{2}} dx$ e) $\int \sin(ax+b)\, dx$ f) $\int \frac{dx}{5x+4}$

Rješenje

a) $\int \frac{dx}{\sqrt[3]{5x}} = \int (5x)^{-\frac{1}{3}} = 5^{-\frac{1}{3}} \int x^{-\frac{1}{3}} dx = \frac{1}{\sqrt[3]{5}} \cdot \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c =$

$= \frac{3}{2\sqrt[3]{5}} x^{\frac{2}{3}} + c = \frac{3}{2\sqrt[3]{5}} \sqrt[3]{x^2} + c$

Koristili smo formulu $\int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1}$ pri čemu je $u = x$, $\alpha = -\frac{1}{3}$.

b) $\int \frac{dt}{\sqrt{3-4t^2}} = \int \frac{dt}{\sqrt{4(\frac{3}{4}-t^2)}} = \frac{1}{\sqrt{4}} \int \frac{dt}{\sqrt{\frac{3}{4}-t^2}} = \frac{1}{2} \cdot \arcsin \frac{t}{\sqrt{\frac{3}{4}}} + c$

$= \frac{1}{2} \arcsin \frac{2t}{\sqrt{3}} + c$

Koristili smo formulu $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + c$ gdje $u = t$, $a = \frac{\sqrt{3}}{2}$

Ovo smo mogli uraditi i na drugi način

$\int \frac{dt}{\sqrt{3-4t^2}} = \int \frac{dt}{\sqrt{3-(2t)^2}} = \left| \begin{array}{l} d(2t) = 2\,dt \\ dt = \frac{1}{2} d(2t) \end{array} \right| = \frac{1}{2} \int \frac{d(2t)}{\sqrt{3-(2t)^2}}$

$= \frac{1}{2} \arcsin \frac{2t}{\sqrt{3}} + c$

Koristili smo istu formulu gdje je $u = 2t$, $a = \sqrt{3}$.

c) $\int \cos 3\varphi \, d\varphi = \left| \begin{array}{l} d(3\varphi) = 3\,d\varphi \\ d\varphi = \frac{1}{3} d(3\varphi) \end{array} \right| = \frac{1}{3} \int \cos 3\varphi \, d(3\varphi) = \frac{1}{3} \sin 3\varphi + C$

Koristili smo formulu $\int \cos u \, du = \sin u + c$ gdje je $u = 3\varphi$.

d) $\int e^{-\frac{x}{2}} dx = \left| \begin{array}{l} d(-\frac{x}{2}) = -\frac{1}{2} dx \\ dx = -2\, d(-\frac{x}{2}) \end{array} \right| = -2 \int e^{-\frac{x}{2}} d(-\frac{x}{2}) = -2 e^{-\frac{x}{2}} + c$

Koristili smo formulu $\int e^u du = e^u + c$ gdje $u = -\frac{x}{2}$.

e) $\int \sin(ax+b)\, dx = \left| \begin{array}{l} d(ax+b) = a\,dx \\ dx = \frac{1}{a} d(ax+b) \end{array} \right| = \frac{1}{a} \int \sin(ax+b)\, d(ax+b) =$

$= -\frac{1}{a} \cos(ax+b) + c$

Koristili smo formulu $\int \sin u \, du = -\cos u + c$ pri čemu je $u = ax+b$

f) $\int \frac{dx}{5x+4} = \left| \begin{array}{l} d(5x+4) = 5\,dx \\ dx = \frac{1}{5} d(5x+4) \end{array} \right| = \frac{1}{5} \int \frac{d(5x+4)}{5x+4} = \frac{1}{5} \ln|5x+4| + c$

Koristili smo formulu $\int \frac{du}{u} = \ln|u| + c$ pri čemu je $u = 5x+4$.

Odrediti integrale

a) $\int (3-2x)^7 dx$ b) $\int \frac{dx}{\cos^2(m-nx)}$ c) $\int \operatorname{tg}\varphi \, d\varphi$

Rj.

a) $\int (3-2x)^7 dx = \left| \begin{array}{l} d(3-2x) = -2\,dx \\ dx = -\frac{1}{2}\,d(3-2x) \end{array} \right| = -\frac{1}{2}\int (3-2x)^7 d(3-2x)$

$= -\frac{1}{2}\cdot\frac{(3-2x)^8}{8} + C = -\frac{1}{16}(3-2x)^8 + C$

Koristili smo formulu $\int u^\alpha du = \frac{x^{\alpha+1}}{\alpha+1} + C$ pri čemu je $u = 3-2x$, $\alpha = 7$.

b) $\int \frac{dx}{\cos^2(m-nx)} = \left| \begin{array}{l} d(m-nx) = -n\,dx \\ dx = \frac{-1}{n}\,d(m-nx) \end{array} \right| = -\frac{1}{n}\int \frac{d(m-nx)}{\cos^2(m-nx)} =$

$= -\frac{1}{n}\operatorname{tg}(m-nx) + C$

Koristili smo formulu $\int \frac{du}{\cos^2 u} = \operatorname{tg} u + C$ gdje $u = m-nx$.

c) $\int \operatorname{tg}\varphi\, d\varphi = \int \frac{\sin\varphi}{\cos\varphi}\, d\varphi = \left| \begin{array}{l} d(\cos\varphi) = -\sin\varphi\, d\varphi \\ \sin\varphi\, d\varphi = -\,d(\cos\varphi) \end{array} \right| = -\int \frac{d(\cos\varphi)}{\cos\varphi}$

$= -\ln|\cos\varphi| + C$

Koristili smo formulu $\int \frac{du}{u} = \ln|u| + C$ gdje $u = \cos\varphi$.

Zadaci za vježbu

Odrediti sljedeće integrale

1. $\int x^4 dx$
2. $\int \sqrt[5]{t^2}\, dt$
3. $\int \frac{dy}{3y^2}$
4. $\int \frac{dx}{x+3}$
5. $\int (\alpha-5)^8 d\alpha$
6. $\int \frac{dx}{x^2+9}$
7. $\int \frac{dv}{\sqrt{v^2+7}}$
8. $\int \frac{dz}{2z^2-4}$
9. $\int \frac{dx}{\sqrt{4-x^2}}$
10. $\int \sin\frac{x}{3}\, dx$
11. $\int \frac{1}{\sin^2 2\varphi}\, d\varphi$
12. $\int e^{4x} dx$
13. $\int \frac{3\,dt}{5^{2t}}$
14. $\int \frac{dx}{2x+5}$
15. $\int \frac{dx}{(3x+2)^3}$
16. $\int \operatorname{ctg} x\, dx$

Rješenja:

1. $\frac{x^5}{5}$ 2. $\frac{5}{7}\sqrt[5]{t^7}$ 3. $-\frac{1}{3y}$ 4. $\ln|x+3|$ 5. $\frac{(\alpha-5)^9}{9}$

6. $\frac{1}{3}\operatorname{arctg}\frac{x}{3}$ 7. $\ln(v+\sqrt{v^2+7})$ 8. $\frac{1}{4\sqrt{2}}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right|$.

9. $\arcsin\frac{x}{2}$ 10. $-3\cos\frac{x}{3}$ 11. $-\frac{1}{2}\operatorname{ctg}2\varphi$ 12. $\frac{1}{4}e^{4x}$

13. $-\frac{3\cdot 5^{-2t}}{2\ln 5}$ 14. $\frac{\ln|2x+5|}{2}$ 15. $-\frac{1}{6(3x+2)^2}$ 16. $\ln|\sin x|$

Izabrani **Zadaci za vježbu** sa rješenjima
(iz lekcije Osnovne formule integriranja)

1. Pomoću osnovnih tabličnih integrala i najjednostavnijih pravila integracije odrediti sljedeće integrale:

a) $\int \sqrt{x}\,dx$

Rj. $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{3}{2}} + C = \frac{2}{3}x^{\frac{3}{2}} + C = \frac{2}{3}\sqrt{x^3} + C = \frac{2}{3}x\sqrt{x} + C$

b) $\int \sqrt[m]{x^n}\,dx$

Rj. $\int \sqrt[m]{x^n}\,dx = \int x^{\frac{n}{m}}\,dx = \frac{x^{\frac{n}{m}+1}}{\frac{m+n}{m}} + C = \frac{m}{m+n}\,x^{\frac{m+n}{m}} + C = \frac{m}{m+n}\sqrt[m]{x^{m+n}} + C =$

$= \frac{m}{m+n}\sqrt[m]{x^m \cdot x^n} + C = \frac{m}{m+n}\,x\sqrt[m]{x^n}$

c) $\int \frac{dx}{x^2}$

Rj. $\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-2+1}}{-1} + C = -x^{-1} + C = -\frac{1}{x} + C = C - \frac{1}{x}$

d) $\int 10^x\,dx$

Rj. $\int 10^x\,dx = \frac{10^x}{\ln 10} + C \approx 0{,}43429\,10^x + C$ $\quad \ln 10 \approx 2{,}30258$

e) $\int a^x e^x\,dx$

Rj. $\int a^x e^x\,dx = \int (a\cdot e)^x\,dx = \frac{(a\cdot e)^x}{\ln(a\cdot e)} + C = \frac{a^x e^x}{\ln a + \ln e} + C = \frac{a^x e^x}{1+\ln a} + C$

f) $\int \frac{dx}{2\sqrt{x}}$

Rj. $\int \frac{dx}{2\sqrt{x}} = \frac{1}{2}\int x^{-\frac{1}{2}}\,dx = \frac{1}{2}\cdot\frac{x^{-\frac{1}{2}+1}}{\frac{1}{2}} + C = \frac{1}{2}\cdot 2\cdot x^{\frac{1}{2}} + C = \sqrt{x} + C$

g) $\int \frac{dh}{\sqrt{2gh}}$

Rj. $\int \frac{dh}{\sqrt{2gh}} = \int \frac{dh}{\sqrt{2g}\cdot\sqrt{h}} = \frac{1}{\sqrt{2g}}\int \frac{dh}{h^{\frac{1}{2}}} = \frac{1}{\sqrt{2g}}\int h^{-\frac{1}{2}}\,dh = \frac{1}{\sqrt{2g}}\cdot\frac{h^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \frac{2}{\sqrt{2g}}\sqrt{h} + C = \sqrt{\frac{4h}{2g}} + C = \sqrt{\frac{2h}{g}} + C$

h) $\int 3{,}4\,x^{-0{,}17}\,dx$

Rj. $\int 3{,}4\,x^{-0{,}17}\,dx = 3{,}4\,\frac{x^{-0{,}17+1}}{0{,}83} + C = \frac{3{,}4}{0{,}83}\,x^{0{,}83} + C \approx 4{,}096\,x^{0{,}83} + C$

i) $\int (1-2u)\,du = \int du - 2\int u\,du$

Rj. $\int (1-2u)\,du = \int du - \int 2u\,du = u - 2\cdot\frac{u^2}{2} + C = u - u^2 + C$

j) $\int (\sqrt{x}+1)(x-\sqrt{x}+1)\,dx$

Rj. $\int (\sqrt{x}+1)(x-\sqrt{x}+1)\,dx = \int (x\sqrt{x} - x + \sqrt{x} + x - \sqrt{x} + 1)\,dx =$

$= \int (x^{\frac{3}{2}} + 1)\,dx = \frac{x^{\frac{3}{2}+1}}{\frac{5}{2}} + x + C = \frac{2}{5}\sqrt{x^5} + x + C = \frac{2}{5}x^2\sqrt{x} + x + C$

k) $\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3}\,dx$ $\qquad$ Rj. $C - \frac{2}{3x\sqrt{x}} - e^x + \ln|x|$

2. Integriranje pomoću razlaganja podintegralne f-je na dijelove

Ako podintegralna f-ja predstavlja algebarsku sumu nekoliko članova, tada, prema osobini IV ($\int [f_1(x)+f_2(x)-f_3(x)]\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx - \int f_3(x)\,dx$) možemo integrirati svaki član posebno.

Koristeći ovo, mnoge integrale možemo svesti na sumu jednostavnijih integrala.

Odrediti integrale

a) $\int (3x^2-2x+5)\,dx$ b) $\int \frac{2x^2+x-1}{x^3}\,dx$ c) $\int (1+e^x)^2\,dx$

d) $\int \frac{2x+3}{x^2-5}\,dx$ e) $\int \frac{x^2}{x^2+1}\,dx$ f) $\int tg^2\varphi\,d\varphi$.

Rj:

a) $\int (3x^2-2x+5)\,dx = \int 3x^2\,dx - \int 2x\,dx + \int 5\,dx =$

$= 3\int x^2dx - 2\int x\,dx + 5\int dx = 3\,\frac{x^3}{3} - 2\cdot\frac{x^2}{2} + 5x + c$

$= x^3 - x^2 + 5x + c$

b) $\int \frac{2x^2+x-1}{x^3}\,dx = 2\int \frac{dx}{x} + \int x^{-2}dx - \int x^{-3}dx =$

$= 2\ln|x| + \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + c = 2\ln|x| - \frac{1}{x} - \frac{1}{2x^2} + c$

c) $\int (1+e^x)^2\,dx = \int (1+2e^x+(e^x)^2)\,dx = \int dx + 2\int e^x dx + \int e^{2x}dx$

$= \left| \begin{array}{l} d(2x) = 2\,dx \\ dx = \frac{1}{2}\,d(2x) \end{array} \right| = \int dx + 2\int e^x dx + \frac{1}{2}\int e^{2x} d(2x) =$

$= x + 2e^x + \frac{1}{2}\,e^{2x} + c$

d) $\int \frac{2x+3}{x^2-5}\,dx = \int \frac{2x}{x^2-5}\,dx + 3\int \frac{dx}{x^2-5} = \int \frac{d(x^2-5)}{x^2-5} + 3\int \frac{dx}{x^2-5}$

$= \ln|x^2-5| + \frac{3}{2\sqrt{5}}\ln\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + c$

e) $\displaystyle\int \frac{x^2}{x^2+1}\,dx = \int \frac{(x^2+1)-1}{x^2+1}\,dx = \int\left(1 - \frac{1}{x^2+1}\right)dx =$

$\displaystyle= \int dx - \int \frac{dx}{x^2+1} = x - \operatorname{arctg} x + c$

f) $\displaystyle\int tg^2\varphi\, d\varphi = \int (tg\varphi)^2\, d\varphi = \int\left(\frac{\sin\varphi}{\cos\varphi}\right)^2 d\varphi =$

$\displaystyle= \int \frac{\sin^2\varphi}{\cos^2\varphi}\,d\varphi = \int \frac{1-\cos^2\varphi}{\cos^2\varphi}\,d\varphi = \int\left(\frac{1}{\cos^2\varphi} - 1\right)d\varphi$

$= tg\varphi - \varphi + c\,.$

Zadaci za vježbu

Odrediti integrale

1. $\displaystyle\int\left(2\sqrt[5]{x} - \sqrt[3]{2x} + 5\right)dx$

2. $\displaystyle\int(\sin\varphi - \cos\varphi)^2\, d\varphi$

3. ** $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx$

4. $\displaystyle\int \frac{5x^2-6x+1}{\sqrt{x}}\,dx$

5. ** $\displaystyle\int \frac{x^3}{x^2+6}\,dx$

6. $\displaystyle\int(tg\,x + ctg\,x)^2\,dx$

7. $\displaystyle\int(e^x - e^{-x})^3\,dx$

8. * $\displaystyle\int \frac{x^2-2}{x+2}\,dx$

* Racionalni algebarski razlomak nazivamo svodljiv, ako je stepen polinoma u brojniku veći ili jednak stepenu polinoma u nazivniku.

** Ovdje, kao u rješenju zadatka $\int \frac{x^2}{x^2+1}\,dx$, podintegralni svodljiv razlomak napisati u nesvodljivom obliku.

Rj:

1. $\frac{5}{3}x\sqrt[5]{x} - \frac{3\sqrt[3]{2}}{4}x\sqrt[3]{x} + 5x$ 2. $\varphi + \frac{1}{2}\cos 2\varphi$

3. $x + \ln\left|\frac{x-1}{x+1}\right|$ 4. $2\sqrt{x}\,(x-1)^3$ 5. $\frac{x^2}{2} - 3\ln(x^2+6)$

6. $tg\,x - ctg\,x$ 7. $\frac{1}{2}(e^{2x} - e^{-2x}) - 2x$

8. $\frac{(x-2)^2}{2} + 2\ln|x+2|$.

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija pomoću razlaganja podintegralne funkcije na dijelove)

1. Pomoću osnovnih tabličnih integrala i najjednostavnijih pravila integracije odrediti sljedeće integrale:

a) $\displaystyle\int \frac{\sqrt{x}-x^3e^x+x^2}{x^3}\,dx$

Rj. $\displaystyle\int \frac{\sqrt{x}-x^3e^x+x^2}{x^3}\,dx=\int\left(\frac{\sqrt{x}}{x^3}-e^x+\frac{1}{x}\right)dx=\int\left(x^{-\frac{5}{2}}-e^x+\frac{1}{x}\right)dx$

$\displaystyle=\frac{x^{-\frac{5}{2}+1}}{-\frac{3}{2}}-e^x+\ln|x|+C=-\frac{2}{3}\cdot\frac{1}{\sqrt{x^3}}-e^x+\ln|x|+C=$

$\displaystyle=C-\frac{2}{3x\sqrt{x}}-e^x+\ln|x|$

b) $\displaystyle\int(2x^{-1,2}+3x^{-0,8}-5x^{0,38})\,dx$

Rj. $\displaystyle\int(2x^{-1,2}+3x^{-0,8}-5x^{0,38})\,dx=2\cdot\frac{x^{-1,2+1}}{-0,2}+3\cdot\frac{x^{-0,8+1}}{0,2}-5\cdot\frac{x^{0,38+1}}{1,38}+C$

$\displaystyle=-\frac{2}{0,2}x^{-0,2}+\frac{3}{0,2}x^{0,2}-\frac{5}{1,38}x^{1,38}+C=C-10x^{-0,2}+15x^{0,2}-3,62x^{1,38}$

c) $\displaystyle\int\left(\frac{1-z}{z}\right)^2dz$

Rj. $\displaystyle\int\left(\frac{1-z}{z}\right)^2dz=\int\left(\frac{1}{z}-1\right)^2dz=\int\left(\frac{1}{z^2}-\frac{2}{z}+1\right)dz=\int\left(z^{-2}-2\,\frac{1}{z}+1\right)dz=$

$\displaystyle=\frac{z^{-1}}{-1}-2\ln|z|+z+C=C-\frac{1}{z}-\ln z^2+z$

d) $\displaystyle\int\frac{(1-x)^2}{x\sqrt{x}}\,dx$

Rj. $\displaystyle\int\frac{(1-x)^2}{x\sqrt{x}}\,dx=\int\frac{1-2x+x^2}{x^{\frac{3}{2}}}\,dx=\int\left(x^{-\frac{3}{2}}-2x^{-\frac{1}{2}}+x^{\frac{1}{2}}\right)dx=$

$\displaystyle=\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}-2\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+C=-2\cdot\frac{1}{x^{\frac{1}{2}}}-2\cdot2\sqrt{x}+\frac{2}{3}\sqrt{x^3}+C=\frac{-2}{\sqrt{x}}-4\sqrt{x}+\frac{2}{3}x\sqrt{x}+C$

$\displaystyle=\frac{-2\cdot3-4\sqrt{x}\cdot3\sqrt{x}+2x\sqrt{x}\cdot\sqrt{x}}{3\sqrt{x}}+C=\frac{2x^2-12x-6}{3\sqrt{x}}+C$

e) $\displaystyle\int\frac{(1+\sqrt{x})^3}{\sqrt[3]{x}}\,dx$

$\frac{1}{2}-\frac{1}{9}=\frac{9-2}{18}=\frac{7}{18}$ $\quad$ $\frac{7}{18}-\frac{4}{18}=\frac{3}{18}=\frac{1}{6}$ $\quad$ $\frac{12}{18}=\frac{2}{3}$

Rj. $\displaystyle\int\frac{(1+\sqrt{x})^3}{\sqrt[3]{x}}\,dx=\int\left(\frac{1+\sqrt{x}}{x^{\frac{1}{9}}}\right)^3dx=\int\left(x^{-\frac{1}{9}}+x^{\frac{7}{18}}\right)^3dx=\int\left(x^{-\frac{1}{3}}+3\cdot x^{-\frac{2}{9}}\cdot x^{\frac{7}{18}}+3x^{-\frac{1}{9}}\cdot x^{\frac{14}{18}}+\right.$

$\displaystyle\left.+x^{\frac{7}{6}}\right)dx=\int\left(x^{-\frac{1}{3}}+3x^{\frac{1}{6}}+3x^{\frac{2}{3}}+x^{\frac{7}{6}}\right)dx=\frac{x^{\frac{2}{3}}}{\frac{2}{3}}+3\cdot\frac{x^{\frac{7}{6}}}{\frac{7}{6}}+3\,\frac{x^{\frac{5}{3}}}{\frac{5}{3}}+\frac{x^{\frac{13}{6}}}{\frac{13}{6}}+C$

$\displaystyle=\frac{3}{2}\sqrt[3]{x^2}+\frac{18}{7}\sqrt[6]{x^7}+\frac{9}{5}\sqrt[3]{x^5}+\frac{6}{13}\sqrt[6]{x^{13}}+C=$

$\displaystyle=\frac{3}{2}\sqrt[3]{x^2}+\frac{18}{7}x\sqrt[6]{x}+\frac{9}{5}x\sqrt[3]{x^2}+\frac{6}{13}x^2\sqrt[6]{x}+C$

f) $\displaystyle\int\frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}\,dx$

$\frac{2}{3}-\frac{1}{2}=\frac{4-3}{6}=\frac{1}{6}$ $\quad$ $\frac{1}{4}-\frac{1}{2}=\frac{1-2}{4}=-\frac{1}{4}$

Rj. $\displaystyle\int\frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}\,dx=\int\frac{x^{\frac{2}{3}}-x^{\frac{1}{4}}}{x^{\frac{1}{2}}}\,dx=\int\left(x^{\frac{1}{6}}-x^{-\frac{1}{4}}\right)dx=\frac{x^{\frac{7}{6}}}{\frac{7}{6}}-\frac{x^{\frac{3}{4}}}{\frac{3}{4}}+C$

$\displaystyle=\frac{6}{7}\sqrt[6]{x^7}-\frac{4}{3}\sqrt[4]{x^3}+C=\frac{6}{7}x\sqrt[6]{x}-\frac{4}{3}\sqrt[4]{x^3}+C$

g) $\displaystyle\int\frac{dx}{\sqrt{3-3x^2}}$

Rj. $\displaystyle\int\frac{dx}{\sqrt{3-3x^2}}=\int\frac{dx}{\sqrt{3(1-x^2)}}=\int\frac{dx}{\sqrt{3}\cdot\sqrt{1-x^2}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{1-x^2}}=\frac{\sqrt{3}}{3}\arcsin x+C$

h) $\displaystyle\int\frac{3\cdot2^x-2\cdot3^x}{2^x}\,dx$

Rj. $\displaystyle\int\frac{3\cdot2^x-2\cdot3^x}{2^x}\,dx=\int\left(3-2\left(\frac{3}{2}\right)^x\right)dx=3x-2\cdot\frac{\left(\frac{3}{2}\right)^x}{\ln\frac{3}{2}}+C=3x-\frac{2\cdot1,5^x}{\ln1,5}+C$

i) $\displaystyle\int\frac{1+\cos^2x}{1+\cos2x}\,dx$

Rj. $\displaystyle\int\frac{1+\cos^2x}{1+\cos2x}\,dx=\int\frac{1+\cos^2x}{\underbrace{\sin^2x+\cos^2x}_{1}+\underbrace{\cos^2x-\sin^2x}_{\cos2x}}\,dx=\int\frac{1+\cos^2x}{2\cos^2x}\,dx$

$\displaystyle=\int\left(\frac{1}{2\cos^2x}+\frac{\cos^2x}{2\cos^2x}\right)dx=\frac{1}{2}\int\left(\frac{1}{\cos^2x}+1\right)dx=\frac{1}{2}\operatorname{tg}x+\frac{1}{2}x+C$

2. Pomoću osnovnih tabličnih integrala i najjednostavnijih pravila integracije odrediti sljedeće integrale:

a) $\displaystyle\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\,dx$

Rj. $\displaystyle\int \frac{\cos 2x}{\cos^2 x\cdot \sin^2 x}\,dx = \int \frac{\cos^2 x-\sin^2 x}{\cos^2 x\cdot\sin^2 x}\,dx = \int\left(\frac{1}{\sin^2 x}-\frac{1}{\cos^2 x}\right)dx =$

$= -\mathrm{ctg}\,x - \mathrm{tg}\,x + C = C - \mathrm{ctg}\,x - \mathrm{tg}\,x$

b) $\displaystyle\int \mathrm{tg}^2 x\,dx$

Rj. $\displaystyle\int \mathrm{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int\left(\frac{1}{\cos^2 x}-1\right)dx = \mathrm{tg}\,x - x + C$

c) $\displaystyle\int \mathrm{ctg}^2 x\,dx$

Rj. $\displaystyle\int \mathrm{ctg}^2 x\,dx = \int\left(\frac{\cos x}{\sin x}\right)^2 dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx = \int\left(\frac{1}{\sin^2 x}-1\right)dx = C - \mathrm{ctg}\,x - x$

d) $\displaystyle\int 2\sin^2 \frac{x}{2}\,dx$

$\left.\begin{array}{l} 1=\cos^2\frac{x}{2}+\sin^2\frac{x}{2} \\ \cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2}\end{array}\right\} \Rightarrow 1-\cos x = 2\sin^2\frac{x}{2}$

Rj. $\displaystyle\int 2\sin^2\frac{x}{2}\,dx = \int (1-\cos x)\,dx = x - \sin x + C$

e) $\displaystyle\int \frac{(1+2x^2)\,dx}{x^2(1+x^2)}$

Rj. $\displaystyle\int \frac{(1+2x^2)}{x^2(1+x^2)}\,dx = \int \frac{1+x^2+x^2}{x^2(1+x^2)}\,dx = \int\left[\frac{1+x^2}{x^2(1+x^2)}+\frac{x^2}{x^2(1+x^2)}\right]dx = \int\left(\frac{1}{x^2}+\frac{1}{1+x^2}\right)dx =$

$= \dfrac{x^{-1}}{-1} + \mathrm{arctg}\,x + C = C - \dfrac{1}{x} + \mathrm{arctg}\,x$

f) $\displaystyle\int \frac{(1+x)^2\,dx}{x(1+x^2)}$

Rj. $\displaystyle\int \frac{(1+x)^2\,dx}{x(1+x^2)} = \int \frac{1+2x+x^2}{x(1+x^2)}\,dx = \int\left(\frac{1+x^2}{x(1+x^2)}+\frac{2x}{x(1+x^2)}\right)dx = \int\left(\frac{1}{x}+\frac{2}{1+x^2}\right)dx =$

$= \ln|x| + 2\,\mathrm{arctg}\,x + C$

g) $\displaystyle\int \frac{dx}{\cos 2x + \sin^2 x}$

Rj. $\displaystyle\int \frac{dx}{\cos 2x+\sin^2 x} = \int \frac{dx}{\cos^2 x-\sin^2 x+\sin^2 x} = \int \frac{dx}{\cos^2 x} = \mathrm{tg}\,x + C$

h) $\displaystyle\int (\arcsin x + \arccos x)\,dx$

Rj. $\displaystyle\int (\arcsin x + \arccos x)\,dx = \int\left(\alpha-\alpha+\frac{\pi}{2}\right)dx = \frac{\pi}{2}x + C$

$\sin\alpha = x$, α je neki ugao $\Rightarrow \arcsin x + \arccos x = \dfrac{\pi}{2}$

$x\in[-1,1]$

$\arcsin x = \alpha$

$\cos\left(-\alpha+\frac{\pi}{2}\right) = \cos(-\alpha)\cos\frac{\pi}{2} - \sin(-\alpha)\sin\frac{\pi}{2} = -\sin(-\alpha) = \sin\alpha$

tj. $\cos\left(-\alpha+\frac{\pi}{2}\right) = \sin\alpha = x \Rightarrow \arccos x = -\alpha + \dfrac{\pi}{2}$

Odredite sljedeće integrale:

3. $\int(2x^3+5x^2-7x-6)dx = \int 2x^3dx + \int 5x^2dx - \int 7x\,dx - \int 6\,dx =$

$= 2\int x^3dx + 5\int x^2dx - 7\int x\,dx - 6\int dx = 2\cdot\frac{x^4}{4} + 5\cdot\frac{x^3}{3} - 7\cdot\frac{x^2}{2} - 6x + C$

$= \frac{x^4}{2} + \frac{5x^3}{3} - \frac{7x^2}{2} - 6x + C$

4. $\int \frac{5x^7+2x^5-x+6}{x^3}dx = \int\left(\frac{5x^7}{x^3}+\frac{2x^5}{x^3}-\frac{x}{x^3}+\frac{6}{x^3}\right)dx =$

$5\int x^4dx + 2\int x^2dx - \int x^{-2}dx + 6\int x^{-3}dx = 5\cdot\frac{x^5}{5}+2\cdot\frac{x^3}{3}-\frac{x^{-1}}{-1}+6\cdot\frac{x^{-2}}{-2}+C$

$= x^5+\frac{2x^3}{3}+\frac{1}{x}-3\cdot\frac{1}{x^2}+C$

5. $\int\sqrt{x\sqrt[3]{x\sqrt{x}}}\,dx = \int\sqrt{x\sqrt[3]{\sqrt{x^2\cdot x}}}\,dx = \int\sqrt{x\sqrt[6]{x^3}}\,dx$

$=\int\sqrt{\sqrt[6]{x^6\cdot x^3}}\,dx = \int\sqrt[12]{x^9}\,dx = \int x^{\frac{9}{12}}dx = \int x^{\frac34}dx$

$= \frac{x^{\frac74}}{\frac74}+C = \frac47\sqrt[4]{x^7}+C$

6. $\int(x^2+\sqrt{x})^2dx = \int(x^4+2x^2\sqrt{x}+x)\,dx = \int x^4dx + 2\int \underbrace{x^2\cdot x^{\frac12}}_{x^{\frac52}}dx + \int x\,dx$

$= \frac{x^5}{5}+2\cdot\frac{x^{\frac72}}{\frac72}+\frac{x^2}{2}+C = \frac{x^5}{5}+\frac47\sqrt{x^7}+\frac{x^2}{2}+C$

7. $\int\frac{x}{x+1}dx = \int\frac{x+1-1}{x+1}dx = \int\left(1-\frac{1}{x+1}\right)dx =$

$= \int dx - \int\frac{dx}{x+1} = x - \ln|x+1| + C$

8. $\int\frac{x^2}{x-1}dx = \int\frac{x^2-1+1}{x-1}dx = \int\frac{x^2-1}{x-1}dx + \int\frac{1}{x-1}dx =$

$= \int\frac{(x-1)(x+1)}{(x-1)}dx + \int\frac{1}{x-1}dx = \int(x+1)\,dx + \int\frac{1}{x-1}dx = \frac{x^2}{2}+x+\ln|x-1|+C$

9. $\int\frac{x^2}{x+2}dx = \int\frac{x^2+4-4}{x+2}dx = \int\left(\frac{x^2-4}{x+2}+\frac{4}{x+2}\right)dx =$

$= \int\frac{(x-2)(x+2)}{x+2}dx + 4\int\frac{dx}{x+2} = \int(x-2)\,dx + 4\int\frac{dx}{x+2} = \frac{x^2}{2}-2x+4\ln|x+2|+C$

II način bi bio da podjelimo x^2 sa $x+2$ pa izvadimo integral od dobijenog rezultata $x^2:(x+2) = x-2+\frac{4}{x+2}$ $\quad \int\frac{x^2}{x+2}dx = \int\left(x-2+\frac{4}{x+2}\right)dx$

10. $\int\frac{x^3}{x-3}dx = \int\frac{x^3-27+27}{x-3}dx = \int\frac{x^3-27}{x-3}dx + \int\frac{27}{x-3}dx =$

$= \int\frac{(x-3)(x^2+3x+9)}{x-3}dx + \int\frac{27}{x-3}dx = \int(x^2+3x+9)\,dx + 27\int\frac{dx}{x-3}$

$= \frac{x^3}{3}+\frac{3x^2}{2}+9x+27\ln|x-3|+C$

11. $\int tg^2x\,dx = \int\frac{\sin^2x}{\cos^2x}dx = \int\frac{1-\cos^2x}{\cos^2x}dx = \int\frac{1}{\cos^2x}dx - \int dx$

$= tg\,x - x + C$

12. $\int\frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}dx$ Rj. $-\frac{24}{17}\sqrt[12]{x^{17}}+\frac45\sqrt[4]{x^5}+\frac43\sqrt[4]{x^3}+C$

13. $\int\frac{e^{3x}+1}{e^x+1}dx$ Rj. $\frac12e^{2x}-e^x+x+C$

14. $\int\frac{1}{\sin^2 2x}dx$ Rj. $-\frac12\cdot\frac{\cos 2x}{\sin 2x}+C$

3. Integracija pomoću smjene promjenjivih

Veoma efikasna metoda integriranja je metoda pomoću smjene promjenjivih, a rezultat metode je da se dati integral zamjeni drugim integralom.

Posmatrajmo dati integral $\int f(x)\,dx$. Ako je moguće, želimo promjenjivu x zamjeniti nekom novom promjenjivom t, koristeći smjenu $x=\varphi(t)$. Tada je $dx=\varphi'(t)\,dt$ pa imamo

$$\int f(x)\,dx=\int f[\varphi(t)]\,\varphi'(t)\,dt=\int F(t)\,dt.$$

Na ovaj način se često zadani integral svodi na elementarni tablični integral. Na primjer, želimo odrediti integral $J=\int\frac{dx}{1+\sqrt{x}}$, uvođenjem smjene $x=t^2$. Tada je $dx=2t\,dt$ pa imamo

$$J=\int\frac{2t\,dt}{1+t}=2\int\frac{t+1-1}{t+1}dt=2\int\left(1-\frac{1}{t+1}\right)dt=$$

$$=2\int dt-2\int\frac{dt}{t+1}=2t-2\ln(t+1)+C=$$

$$=2\sqrt{x}-2\ln(\sqrt{x}+1)+C$$

Isti integral smo mogli odrediti i uvođenjem smjene $t=1+\sqrt{x}$ iz čega slijedi $x=(t-1)^2$, $dx=2(t-1)\,dt$

$$J=\int\frac{2(t-1)\,dt}{t}=2\int\left(1-\frac{1}{t}\right)dt=2t-2\ln t+C=2(1+\sqrt{x})-2\ln(1+\sqrt{x})+C$$

(#) Odrediti integrale

a) $\int\frac{2x\,dx}{x^4+3}$ b) $\int\frac{\sin x\,dx}{\sqrt{1+2\cos x}}$ c) $\int\frac{x\,dx}{\sqrt[3]{x^2+a}}$

d) $\int\frac{\sqrt{1+\ln x}}{x}dx$ e) $\int\frac{dy}{\sqrt{e^y+1}}$ f) $\int\frac{dt}{\sqrt{(1-t^2)^3}}$

Rj.

a) $$\int\frac{2x\,dx}{x^4+3}=\left|\begin{array}{l}x^2=t\\2x\,dx=dt\end{array}\right|=\int\frac{dt}{t^2+3}=\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}}+C=$$

$$=\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x^2}{\sqrt{3}}+C$$

b) $$\int\frac{\sin x\,dx}{\sqrt{1+2\cos x}}=\left|\begin{array}{l}1+2\cos x=t\\-2\sin x\,dx=dt\\\sin x\,dx=-\frac{1}{2}dt\end{array}\right|=-\frac{1}{2}\int\frac{dt}{\sqrt{t}}=-\frac{1}{2}\int t^{-1/2}dt$$

$$=-\frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+C=C-\sqrt{t}=C-\sqrt{1+2\cos x}$$

c) $$\int\frac{x\,dx}{\sqrt[3]{x^2+a}}=\left|\begin{array}{l}x^2+a=z\\2x\,dx=dz\\x\,dx=\frac{1}{2}dz\end{array}\right|=\frac{1}{2}\int\frac{dz}{\sqrt[3]{z}}=\frac{1}{2}\int z^{-\frac{1}{3}}dz=$$

$$=\frac{1}{2}\cdot\frac{z^{\frac{2}{3}}}{\frac{2}{3}}+C=\frac{1}{2}\cdot\frac{3}{2}\cdot\sqrt[3]{z^2}+C=\frac{3}{4}\sqrt[3]{(x^2+a)^2}+C$$

d) $$\int\frac{\sqrt{1+\ln x}}{x}dx=\left|\begin{array}{l}1+\ln x=v\\\frac{1}{x}dx=dv\end{array}\right|=\int\sqrt{v}\,dv=\int v^{\frac{1}{2}}dv=\frac{v^{\frac{3}{2}}}{\frac{3}{2}}+C$$

$$=\frac{2}{3}\sqrt{v^3}+C=\frac{2}{3}\sqrt{(1+\ln x)^3}+C$$

e) $\displaystyle\int \frac{dy}{\sqrt{e^y+1}} = \left| \begin{array}{ll} e^y+1=t^2 & (t^2-1)\,dy = 2t\,dt \\ e^y = t^2-1 & dy = \dfrac{2t\,dt}{t^2-1} \\ de^y = d(t^2-1) & \\ e^y dy = 2t\,dt & \end{array} \right| =$

$$= \int \frac{\frac{2t\,dt}{t^2-1}}{\sqrt{t^2}} = \int \frac{2t\,dt}{t(t^2-1)} = 2\int \frac{dt}{t^2-1} = 2\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \ln \frac{\sqrt{e^y+1}-1}{\sqrt{e^y+1}+1} + C$$

f) $\displaystyle\int \frac{dt}{\sqrt{(1-t^2)^3}} = \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \\ 1-t^2 = 1-\sin^2\varphi = \cos^2\varphi \end{array} \right| =$

$$= \int \frac{\cos\varphi\, d\varphi}{\sqrt{\cos^6\varphi}} = \int \frac{\cos\varphi\, d\varphi}{\cos^3\varphi} = \int \frac{d\varphi}{\cos^2\varphi} = \operatorname{tg}\varphi + C$$

$$= \frac{\sin\varphi}{\cos\varphi} + C = \frac{\sin\varphi}{\sqrt{1-\sin^2\varphi}} + C = \frac{t}{\sqrt{1-t^2}} + C$$

Zadaci za vježbu

Izračunati sljedeće integrale i provjeriti rezultat diferenciranjem:

1. $\displaystyle\int \frac{x^2\,dx}{5-x^6}$. Pomoću smjene $t = x^3$.
2. $\displaystyle\int \frac{e^x\,dx}{3+4e^x}$. Pomoću smjene $z = 3+4e^x$.
3. $\displaystyle\int \operatorname{tg}^3\varphi\, d\varphi$. Pomoću smjene $\varphi = \operatorname{arc\,tg} t$.
4. $\displaystyle\int x^3\sqrt{a-x^2}\,dx$. Pomoću smjene $\sqrt{a-x^2} = z$.
5. $\displaystyle\int \frac{x^2-x}{(x-2)^3}\,dx$. Pomoću smjene $x-2 = t$
6. $\displaystyle\int x\sqrt{a-x}\,dx$. Pomoću smjene $a-x = t^2$
7. * $\displaystyle\int \frac{dx}{x\sqrt{1+x^2}}$. Pomoću smjene $x = \dfrac{1}{t}$.
8. * $\displaystyle\int \frac{dx}{\sin 2x}$. Pomoću smjene $\operatorname{tg} x = z$.

Odrediti integrale

9. $\displaystyle\int \frac{x\,dx}{\sqrt{x^4+1}}$
10. $\displaystyle\int \frac{\sqrt{x}\,dx}{1+\sqrt{x}}$
11. $\displaystyle\int \frac{e^{2x}\,dx}{e^x-1}$
12. $\displaystyle\int \frac{dx}{x\ln x}$

13. $$\int \frac{\cos x \, dx}{\sqrt{1+2\sin^2 x}}$$

14. $$\int \frac{\sin 2x \, dx}{\sqrt{2+\cos^2 x}}$$

15.* $$\int \frac{e^{2x} \, dx}{\sqrt[4]{1+e^x}}$$

16.* $$\int \frac{\sqrt{x} \, dx}{1+\sqrt[4]{x^3}}$$

Rješenja:

1. $\frac{1}{6\sqrt{5}} \ln \left| \frac{x^3+\sqrt{5}}{x^3-\sqrt{5}} \right|$

2. $\frac{1}{4} \ln(3+4e^x)$

3. $\frac{1}{2} tg^2 \varphi + \ln|\cos\varphi|$

4. $-\frac{3x^2+2a}{15} \sqrt{(a-x^2)^3}$

5. $\ln|x-2| - \frac{3x-5}{(x-2)^2}$

6. $\frac{2}{15}(3x^2-ax-2a^2)\sqrt{a-x}$

7. $\pm \ln \frac{x}{1 \pm \sqrt{1+x^2}}$, gdje je "+" ako $x>0$, "−" ako je $x<0$

ili drugačije $\ln \frac{|x|}{1+}$

8. $\frac{1}{2} \ln|tg x|$

9. $\frac{1}{2} \ln(x^2+\sqrt{1+x^4}$

10. $x-2\sqrt{x}+2\ln(1+\sqrt{x})$

11. $e^x + \ln|e^x-1|$

12. $\ln|\ln x|$

13. $\frac{1}{\sqrt{2}} \ln\left(\sin x + \sqrt{\frac{1}{2}+\sin^2 x}\right)$

14. $-2\sqrt{2+\cos^2 x}$

15. $\frac{4}{21}(3e^x-4)\sqrt[4]{(e^x+1)^3}$

16. $\frac{4}{3}\left(\sqrt[4]{x^3} - \ln\left(1+\sqrt[4]{x^3}\right)\right).$

Izabrani **Zadaci za vježbu** sa rješenjima
(iz lekcije Integracija pomoću zamjene promjenjivih)

1. $$\int \frac{dx}{2x+5} = \left| \begin{array}{l} t=2x+5 \\ dt=2dx \\ dx=\frac{dt}{2} \end{array} \right| = \int \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t|+C = \frac{1}{2}\ln|2x+5|+C$$

2. $$\int \sin(4x+1)\,dx = \left| \begin{array}{l} 4x+1=t \\ 4dx=dt \\ dx=\frac{dt}{4} \end{array} \right| = \int \sin t \cdot \frac{dt}{4} = \frac{1}{4}\int \sin t\,dt = -\frac{1}{4}\cos t + C = -\frac{1}{4}\cos(4x+1)+C$$

3. $$\int (3x-1)^9\,dx = \left| \begin{array}{l} 3x-1=t \\ 3dx=dt \\ dx=\frac{dt}{3} \end{array} \right| = \int t^9 \cdot \frac{dt}{3} = \frac{1}{3}\int t^9\,dt = \frac{1}{3}\cdot\frac{t^{10}}{10}+C = \frac{(3x-1)^{10}}{30}+C$$

4. $$\int e^{1-3x}\,dx = \left| \begin{array}{l} 1-3x=t \\ -3dx=dt \\ dx=-\frac{dt}{3} \end{array} \right| = \int e^t \cdot -\frac{dt}{3} = -\frac{1}{3}\int e^t\,dt = -\frac{1}{3}e^t+C = -\frac{1}{3}e^{1-3x}+C$$

5. $$\int \frac{dx}{\sqrt{1-(3x+2)^2}} = \left| \begin{array}{l} 3x+2=t \\ 3dx=dt \\ dx=\frac{dt}{3} \end{array} \right| = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{3}\arcsin t + C = \frac{1}{3}\arcsin(3x+2)+C$$

6.ˇ $\int \cos(6x+4)\,dx$ Rj. $\frac{1}{6}\sin(6x+4)+C$

7.ˇ $\int \frac{dx}{\cos^2(7x+8)}$ Rj. $\frac{1}{7}\cdot\frac{\sin(7x+8)}{\cos(7x+8)}$

8.ˇ $\int \frac{dx}{1+(5x-2)^2}$ Rj. $\frac{1}{5}\, arctg(5x-2)$

9. $$\int \frac{dx}{4x^2+9} = \int \frac{dx}{(2x)^2+3^2} = \left| \begin{array}{l} 2x = 3t \\ 2dx = 3dt \\ dx = \frac{3}{2}dt \\ t = \frac{2x}{3} \end{array} \right| = \frac{3}{2}\int \frac{dt}{(3t)^2+3^2} = \frac{3}{2}\int \frac{dt}{9t^2+9} =$$
$$= \frac{3}{2}\cdot\frac{1}{9}\int \frac{dt}{t^2+1} = \frac{1}{6}\,\mathrm{arctg}\, t + C = \frac{1}{6}\,\mathrm{arctg}\,\frac{2x}{3} + C$$

10. $$\int \frac{dx}{\sqrt{2x^2+25}} = \int \frac{dx}{\sqrt{(\sqrt{2}x)^2+5^2}} = \left| \begin{array}{l} \sqrt{2}x = 5t \\ \sqrt{2}\,dx = 5dt \\ dx = \frac{5}{\sqrt{2}}\,dt \\ t = \frac{\sqrt{2}}{5}x \end{array} \right| = \frac{5}{\sqrt{2}}\int \frac{dt}{\sqrt{\underbrace{25t^2+25}_{25\cdot(t^2+1)}}} =$$
$$= \frac{5}{\sqrt{2}}\cdot\frac{1}{5}\int \frac{dt}{\sqrt{t^2+1}} = \frac{1}{\sqrt{2}}\cdot \ln\left|t+\sqrt{t^2+1}\right| + C = \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}}{5}x + \sqrt{\frac{2}{25}x^2+1}\right| + C$$

11. $$\int \frac{dx}{5x^2-49} = \int \frac{dx}{(\sqrt{5}x)^2-7^2} = \left| \begin{array}{l} \sqrt{5}x = 7t \\ \sqrt{5}dx = 7dt \\ dx = \frac{7}{\sqrt{5}}\,dt \\ t = \frac{\sqrt{5}x}{7} \end{array} \right| = \frac{7}{\sqrt{5}}\int \frac{dt}{49t^2-49} = \frac{7}{\sqrt{5}}\cdot\frac{1}{49}\int \frac{dt}{t^2-1}$$
$$= \frac{1}{7\sqrt{5}}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{14\sqrt{5}}\ln\left|\frac{\frac{\sqrt{5}}{7}x-1}{\frac{\sqrt{5}}{7}x+1}\right| + C = \frac{1}{14\sqrt{5}}\ln\left|\frac{\sqrt{5}x-7}{\sqrt{5}x+7}\right|$$

12. $$\int \frac{dx}{\sqrt{7-9x^2}} = \int \frac{dx}{\sqrt{(\sqrt{7})^2-(3x)^2}} = \left| \begin{array}{l} 3x = \sqrt{7}t \\ 3dx = \sqrt{7}\,dt \\ dx = \frac{\sqrt{7}}{3}\,dt \\ t = \frac{3x}{\sqrt{7}} \end{array} \right| = \frac{\sqrt{7}}{3}\int \frac{dt}{\sqrt{7-7t^2}} = \frac{\sqrt{7}}{3}\cdot\frac{1}{\sqrt{7}}\int \frac{dt}{\sqrt{1-t^2}}$$
$$= \frac{1}{3}\arcsin t + C = \frac{1}{3}\arcsin\left(\frac{3x}{\sqrt{7}}\right) + C$$

13. $\int \frac{dx}{4x^2+11}$, Rj. $\frac{\sqrt{11}}{22}\,\mathrm{arctg}\,\frac{2\sqrt{11}x}{11} + C$

14. $\int \frac{dx}{\sqrt{9x^2-16}}$, Rj. $\frac{1}{3}\ln\left|3x+\sqrt{9x^2-16}\right| + C$

15. $\int \frac{dx}{\sqrt{2x^2+5}}$, Rj. $\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{10}}{5}x + \sqrt{\frac{2}{5}x^2+1}\right| + C$

16. $$\int \mathrm{ctg}\,x\,dx = \int \frac{\cos x}{\sin x}\,dx = \left| \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C =$$
$$= \ln|\sin x| + C$$

17. $$\int \frac{3x^2+4x-4}{x^3+2x^2-4x+6}\,dx = \left| \begin{array}{l} x^3+2x^2-4x+6 = t \\ (3x^2+4x-4)\,dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C =$$
$$= \ln|x^3+2x^2-4x+6| + C$$

18. $$\int \frac{x-5}{\sqrt{x^2-10x+7}}\,dx = \left| \begin{array}{l} x^2-10x+7 = t \\ (2x-10)\,dx = dt \\ (x-5)\,dx = \frac{dt}{2} \end{array} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C =$$
$$= \frac{1}{2}\cdot 2\sqrt{t} + C = \sqrt{x^2-10x+7} + C$$

19. $$\int \frac{x-3}{x^2-6x+7}\,dx = \left| \begin{array}{l} x^2-6x+7 = t \\ (2x-6)\,dx = dt \\ (x-3)\,dx = \frac{dt}{2} \end{array} \right| = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C$$
$$= \frac{1}{2}\ln|x^2-6x+7| + C$$

20. $$\int \frac{x^3\,dx}{\sqrt{x^4+1}} = \left| \begin{array}{l} x^4+1 = t \\ 4x^3dx = dt \\ x^3dx = \frac{dt}{4} \end{array} \right| = \frac{1}{4}\int \frac{dt}{\sqrt{t}} = \frac{1}{4}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{1}{4}\cdot 2\sqrt{t} + C$$
$$= \frac{1}{2}\sqrt{x^4+1} + C$$

21. $\int \frac{3x^2}{\sqrt{x^3-2}}\,dx$ Rj. $2\sqrt{x^3-2} + C$

22. $\int \mathrm{tg}\,x\,dx$ Rj. $-\ln|\cos x| + C$

23. $\int \frac{\sin x}{\sqrt{5\cos x-2}}\,dx$ Rj. $-\frac{2}{5}\sqrt{5\cos x-2} + C$

24. $$\int e^{\cos x}\cdot \sin x\,dx = \left|\begin{array}{l} \cos x = t \\ -\sin x\,dx = dt \\ \sin x\,dx = -dt \end{array}\right| = \int e^{t}\cdot(-dt) = -\int e^{t}dt = -e^{t}+c = -e^{\cos x}+c$$

25. $$\int \frac{dx}{x\sqrt[5]{\ln x}} = \left|\begin{array}{l} \ln x = t^5 \\ \frac{1}{x}dx = 5t^4dt \\ t = \sqrt[5]{\ln x} \end{array}\right| = \int \frac{5t^4dt}{t} = 5\int t^3dt = 5\cdot\frac{t^4}{4}+c = \frac{5}{4}\sqrt[5]{\ln^4 x}+c$$

26. $$\int \frac{x^3dx}{x^8-2} = \int \frac{x^3\,dx}{(x^4)^2-2} = \left|\begin{array}{l} x^4 = t \\ 4x^3dx = dt \\ x^3dx = \frac{1}{4}dt \end{array}\right| = \frac{1}{4}\int\frac{dt}{t^2-2} = \frac{1}{4}\cdot\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c = \frac{1}{8\sqrt{2}}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right|+c$$

27. $$\int \frac{\sqrt[3]{\operatorname{tg} x}}{\cos^2 x}dx = \left|\begin{array}{l} \operatorname{tg} x = t^3 \\ \frac{1}{\cos^2 x}dx = 3t^2dt \end{array}\right| = \int\sqrt[3]{t^3}\cdot 3t^2dt = 3\int t^3dt = 3\frac{t^4}{4}+c = \frac{3}{4}\sqrt[3]{\operatorname{tg}^4 x}+c$$

28. $$\int x(1-x)^{10}dx = \left|\begin{array}{l} 1-x = t \\ -dx = dt \\ dx = -dt \\ x = 1-t \end{array}\right| = \int(1-t)\,t^{10}\cdot(-dt) = -\int(t^{10}-t^{11})dt$$

$$= -\frac{t^{11}}{11}+\frac{t^{12}}{12}+c = \frac{(1-x)^{12}}{12}-\frac{(1-x)^{11}}{11}+c$$

29. $$\int\frac{x^2-1 \;\;/:x^2}{x^4+1 \;\;/:x^2}dx = \int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \left|\begin{array}{l} \text{treba da stvorimo zamjenu takvu da dobijemo } \left(1-\frac{1}{x^2}\right)dx = dt \\ t = x+\frac{1}{x} \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt \\ \left(x+\frac{1}{x}\right)^2 = t^2 \Rightarrow x^2+2x\cdot\frac{1}{x}+\frac{1}{x^2} = t^2 \\ x^2+\frac{1}{x^2} = t^2-2 \end{array}\right|$$

$$= \int\frac{dt}{t^2-2} = \frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c = \frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+c$$

30. $$\int\sqrt{\frac{\arcsin x}{1-x^2}}dx = \int\frac{\sqrt{\arcsin x}}{\sqrt{1-x^2}}dx = \left|\begin{array}{l} \arcsin x = t^2 \\ \frac{dx}{\sqrt{1-x^2}} = 2t\,dt \\ t = \sqrt{\arcsin x} \end{array}\right| =$$

$$= \int\sqrt{t^2}\,2t\,dt = 2\int t^2dt = 2\cdot\frac{t^3}{3}+c = \frac{2}{3}\sqrt{\arcsin^3 x}+c$$

31. $$\int(x+4)\sqrt[5]{2x-1}\,dx = \left|\begin{array}{ll} 2x-1 = t^5 & 2x = t^5+1 \\ 2\,dx = 5t^4dt & x = \frac{t^5+1}{2} \\ dx = \frac{5}{2}t^4dt & t = \sqrt[5]{2x-1} \end{array}\right| = \int\left(\frac{t^5+1}{2}+4\right)\sqrt[5]{t^5}\cdot\frac{5}{2}t^4dt$$

$$= \frac{5}{2}\int\frac{t^5+1+8}{2}t^5dt = \frac{5}{4}\int(t^5+9)t^5dt = \frac{5}{4}\int(t^{10}+9t^5)dt =$$

$$= \frac{5}{4}\cdot\frac{t^{11}}{11}+\frac{5}{4}\cdot\cancel{9}^{\,3}\cdot\frac{t^6}{\cancel{6}_{\,2}}+c = \frac{5}{44}\sqrt[5]{(2x-1)^{11}}+\frac{15}{8}\sqrt[5]{(2x-1)^6}+c$$

32. ISPITNI
$$\int\frac{\ln x\,dx}{x\sqrt{1+\ln x}} = \left|\begin{array}{l} \ln x = t \\ \frac{1}{x}dx = dt \end{array}\right| = \int\frac{t+1-1}{\sqrt{1+t}}dt = \int\frac{t+1}{\sqrt{1+t}}dt-\int\frac{dt}{\sqrt{1+t}}$$

$$= \int(1+t)^{\frac{1}{2}}dt-\int(1+t)^{-\frac{1}{2}}dt = \frac{(1+t)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(1+t)^{\frac{1}{2}}}{\frac{1}{2}}+c = \frac{2}{3}(1+t)^{\frac{3}{2}}-2(1+t)^{\frac{1}{2}}+c$$

$$= \frac{2}{3}(1+t)^{\frac{1}{2}}\cdot((1+t)-3)+c = \frac{2}{3}\sqrt{1+\ln x}\,(\ln x-2)+c$$

33. ✓ ISPITNI
$$\int\frac{e^{3x}(10-2e^{3x})}{2e^{6x}-10e^{3x}+12}dx$$

Rj. $-\frac{1}{6}\ln\left|e^{6x}-5e^{3x}+6\right|+\frac{5}{6}\ln\left|\frac{e^{3x}-3}{e^{3x}-2}\right|+c$

34. ✓ $$\int\frac{1}{x^2}\sin\frac{1}{x}dx$$

35. ✓ $$\int x(2x+3)^7dx$$

36. ✓ $$\int\frac{\arcsin x}{\sqrt{1-x^2}}dx$$

37. ✓ $$\int\frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}dx$$

(#) Izračunati integral $\int \sqrt{\frac{x-2}{x+2}}\,dx$.

Rj. $\int \sqrt{\frac{x-2}{x+2}}\,dx = \int \frac{\sqrt{x-2}}{\sqrt{x+2}}\,dx = \int \frac{\sqrt{x-2}\cdot\sqrt{x-2}}{\sqrt{x+2}\cdot\sqrt{x-2}}\,dx = \int \frac{x-2}{\sqrt{x^2-4}}\,dx$

$$= \int \frac{x}{\sqrt{x^2-4}}\,dx - 2\int \frac{dx}{\sqrt{x^2-4}}$$

$$\int \frac{x}{\sqrt{x^2-4}}\,dx = \left| \begin{array}{l} x^2-4=t \\ 2x\,dx = dt \\ x\,dx = \frac{1}{2}\,dt \end{array} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\int t^{-\frac{1}{2}}dt = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \sqrt{t} + c = \sqrt{x^2-4} + c$$

$$\int \frac{dx}{\sqrt{x^2-4}} = \left| \begin{array}{l} x = 2s \\ dx = 2\,ds \\ s = \frac{1}{2}x \end{array} \right| = \int \frac{2\,ds}{\sqrt{4s^2-4}} = \frac{2}{\sqrt{4}}\int \frac{ds}{\sqrt{s^2-1}} = \ln|s + \sqrt{s^2-1}| + c_1$$

$$= \ln\left|\frac{1}{2}x + \sqrt{\frac{1}{4}x^2 - 1}\right| + c_1 = \ln\left|\frac{1}{2}x + \frac{1}{2}\sqrt{x^2-4}\right| + c_1$$

$$= \ln\frac{1}{2} + \ln|x + \sqrt{x^2-4}| + c_1 = \ln|x+\sqrt{x^2-4}| + c$$

$$\int \sqrt{\frac{x-2}{x+2}}\,dx = \sqrt{x^2-4} - 2\ln|x+\sqrt{x^2-4}| + c$$

(#) Izračunati integral $\int x^3\sqrt{1+a^2x^2}\,dx$, $(a>0)$.

Rj. $\int x^3\sqrt{1+a^2x^2}\,dx = \int x^2\cdot x\cdot\sqrt{1+a^2x^2}\,dx = \left| \begin{array}{l} 1+a^2x^2 = t^2 \\ a^2\cdot 2x\,dx = 2t\,dt \\ x\,dx = \frac{1}{a^2}\,t\,dt \\ a^2x^2 = t^2-1 \\ x^2 = \frac{1}{a^2}(t^2-1) \end{array} \right|$

$$= \int \frac{1}{a^2}(t^2-1)\cdot\frac{1}{a^2}\,t\cdot t\,dt =$$

$$= \frac{1}{a^4}\int (t^4-t^2)\,dt = \frac{1}{a^4}\cdot\frac{1}{5}t^5 - \frac{1}{a^4}\cdot\frac{1}{3}t^3 =$$

$$= \frac{1}{5a^4}\sqrt{(1+a^2x^2)^5} - \frac{1}{3a^4}\sqrt{(1+a^2x^2)^3} + c$$

(#) Izračunati integral $\int \frac{dx}{\sqrt[4]{x^3}(1+\sqrt[6]{x})}$.

Rj. $\int \frac{dx}{\sqrt[4]{x^3}(1+\sqrt[6]{x})} = \left| \begin{array}{ll} x = t^{12} & t = \sqrt[12]{x} \\ dx = 12t^{11}dt & \\ \sqrt[4]{x^3} = \sqrt[4]{t^{36}} = t^9 & \\ \sqrt[6]{x} = \sqrt[6]{t^{12}} = t^2 & \end{array} \right| = \int \frac{12t^{11}\,dt}{t^9(1+t^2)} =$

$$= 12\int \frac{t^2 \overset{+1-1}{}}{1+t^2}\,dt = 12\int \frac{t^2+1}{t^2+1}\,dt - 12\int \frac{dt}{1+t^2} =$$

$$= 12t - 12\,\mathrm{arctg}\,t + c = 12\sqrt[12]{x} - 12\,\mathrm{arctg}\,\sqrt[12]{x} + c$$

4. Metoda parcijalne integracije

Prema formuli diferenciranja znamo da je

$$d(uv) = u\,dv + v\,du$$

a znamo i da je $\int d(uv) = uv$. Prema tome nije teško vidjeti da vrijedi

$$\int u\,dv = uv - \int v\,du \qquad \ldots (*)$$

Formulu $(*)$ nazivamo formula parcijalne integracije.

Posmatrajmo neki dati integral $\int f(x)g(x)\,dx$. Metodom parcijalne integracije biramo f-ju $u = f(x)$ i $dv = g(x)\,dx$ tako da se integral $\int v\,du$ može jednostavno riješiti.

(#) Odrediti integrale

a) $\int x\cos x\,dx$ b) $\int \frac{\ln x}{x^3}\,dx$ c) $\int x\,\mathrm{arctg}\,x\,dx$

d) $\int \arcsin x\,dx$ e) $\int x^2 e^{3x}\,dx$ f) $\int e^{-x}\cos\frac{x}{2}\,dx$

Rj.

a)
$$\int x\cos x\,dx = \begin{vmatrix} u = x & dv = \cos x\,dx \\ du = dx & v = \int \cos x\,dx = \sin x \end{vmatrix} =$$
$$= x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

b)
$$\int \frac{\ln x}{x^3}\,dx = \begin{vmatrix} u = \ln x & dv = \frac{dx}{x^3} \\ du = \frac{1}{x}\,dx & v = \int \frac{dx}{x^3} = \frac{x^{-2}}{-2} = -\frac{1}{2x^2} \end{vmatrix} =$$
$$= -\frac{\ln x}{2x^2} - \int \frac{(-1)}{2x^2}\cdot\frac{1}{x}\,dx = -\frac{\ln x}{2x^2} + \frac{1}{2}\int \frac{dx}{x^3} =$$
$$= -\frac{1}{2x^2}\ln x + \frac{1}{2}\cdot\frac{x^{-2}}{-2} + C = -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C$$
$$= C - \frac{2\ln x + 1}{4x^2} + C$$

c)
$$\int x\,\mathrm{arctg}\,x\,dx = \begin{vmatrix} u = \mathrm{arctg}\,x & dv = x\,dx \\ du = \frac{dx}{1+x^2} & v = \int x\,dx = \frac{x^2}{2} \end{vmatrix} =$$
$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$ Izračunajmo posebno $\int \frac{x^2}{1+x^2}\,dx$ ...(1)

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int \frac{1+x^2-1}{1+x^2}\,dx =$$

$$= \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \operatorname{arctg} x + c_1$$

Sad, prema (1) imamo

$$I = \frac{1}{2}x^2 \operatorname{arctg} x - \frac{1}{2}x + \frac{1}{2}\operatorname{arctg} x + c =$$

$$= c - \frac{1}{2}x + \frac{x^2+1}{2}\operatorname{arctg} x$$

d) $$\int \arcsin x\,dx = \left| \begin{array}{ll} u = \arcsin x & dv = dx \\ du = \dfrac{dx}{\sqrt{1-x^2}} & v = \int dx = x \end{array} \right| =$$

$$= x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = \left| \begin{array}{l} d(1-x^2) = -2x\,dx \\ x\,dx = -\frac{1}{2}d(1-x^2) \end{array} \right| =$$

$$= x\arcsin x + \frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} = x\arcsin x + \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}} d(1-x^2)$$

$$= x\arcsin x + \frac{1}{2}\,\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = x\arcsin x + \sqrt{1-x^2} + c$$

e) $$\int x^2 e^{3x}\,dx = \left| \begin{array}{ll} u = x^2 & dv = e^{3x}dx \\ du = 2x\,dx & v = \int e^{3x}dx = \frac{1}{3}\int e^{3x}d(3x) = \frac{1}{3}e^{3x} \end{array} \right| =$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x\,e^{3x}dx \quad \ldots (1)$$

$$\int x\,e^{3x}dx = \left| \begin{array}{ll} u = x & dv = e^{3x}dx \\ du = dx & v = \frac{1}{3}e^{3x} \end{array} \right| = \frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x}dx =$$

$$= \frac{1}{3}x e^{3x} - \frac{1}{3}\cdot\frac{1}{3}e^{3x} + c_1 = \frac{1}{3}x e^{3x} - \frac{1}{9}e^{3x} + c_1$$

Prema (1) imamo

$$\int x^2 e^{3x}dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}x e^{3x} - \frac{1}{9}e^{2x} + c_1\right) =$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + C$$

$$= \frac{1}{27}e^{3x}\left(9x^2 - 6x + 2\right) + C$$

f) $$I = \int e^{-x}\cos\frac{x}{2}\,dx = \left| \begin{array}{ll} u = e^{-x} & dv = \cos\frac{x}{2}dx \\ du = e^{-x}\cdot(-1) = -e^{-x}dx & v = \int\cos\frac{x}{2}dx = 2\int\cos\frac{x}{2}\,d\left(\frac{x}{2}\right) \\ & \quad = 2\sin\frac{x}{2} \end{array} \right|$$

$$= 2e^{-x}\sin\frac{x}{2} + 2\int e^{-x}\sin\frac{x}{2}\,dx = \left| \begin{array}{ll} u = e^{-x} & dv = \sin\frac{x}{2}dx \\ du = -e^{-x}dx & v = (-2)\cos\frac{x}{2} \end{array} \right| =$$

$$= 2e^{-x}\sin\frac{x}{2} + 2\left((-2)\,e^{-x}\cos\frac{x}{2} - 2\int e^{-x}\;\; s\frac{x}{2}\,dx\right)$$

$$= 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2} - 4\int e^{-x}\cos\frac{x}{2}\,dx$$

Tj.

$$I = 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2} - 4I$$

$$5I = 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2}$$

$$I = \frac{2}{5}e^{-x}\left(\sin\frac{x}{2} - 2\cos\frac{x}{2}\right) + C$$

Zadaci za vježbu

1. $\int x \sin x \, dx$ 2. $\int x^2 \ln x \, dx$ 3. $\int \ln(x^n) \, dx$

4. $\int (x^2+1) e^{-2x} \, dx$ 5. $\int \frac{x}{\cos^2 x} \, dx$ 6. $\int x \ln(x-1) \, dx$

7. $\int \operatorname{arcctg} t \, dt$ 8. $\int \ln(1+x^2) \, dx$ 9. $\int e^{ax} \sin bx \, dx$

10.* $\int \frac{\arcsin x}{x^2} \, dx$ 11.* $\int \frac{\ln x \, dx}{(x+1)^2}$

12.* $\int \operatorname{arctg} \sqrt{2x-1} \, dx$

Rješenja

1. $\sin x - x \cos x$ 2. $\frac{x^3}{9}(3\ln x - 1)$ 3. $nx(\ln x - 1)$

4. $-\frac{2x^2+2x+3}{4e^{2x}}$ 5. $x \operatorname{tg} x + \ln|\cos x|$ 6. $\frac{x^2-1}{2} \ln|x-1| - \frac{x^2}{4} - \frac{x}{2}$ 7. $t \operatorname{arcctg} t + \frac{1}{2}\ln(1+t^2)$

8. $x\ln(x^2+1) - 2x + 2\operatorname{arctg} x$ 9. $\frac{e^{ax}(a \sin bx - b\cos bx)}{a^2+b^2}$

10. $\ln \frac{1-\sqrt{1-x^2}}{x} - \frac{1}{x}\arcsin x$ 11. $\frac{x\ln x}{x+1} - \ln|x+1|$

12. $x \operatorname{arctg}\sqrt{2x-1} - \frac{1}{2}\sqrt{2x-1}$

Izabrani **Zadaci za vježbu** sa rješenjima (iz lekcije Metoda parcijalne integracije)

1. $$\int x e^x dx = \left| \begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = \int e^x dx = e^x \end{array} \right| = xe^x - \int e^x dx = xe^x - e^x + C$$

Da smo uzeli smjene $u = e^x$, $dv = x\,dx$ dobili bi komplikovan izraz.

2. $$\int x^2 \sin 3x \, dx = \left| \begin{array}{ll} u = x^2 & dv = \sin 3x \, dx \\ du = 2x\,dx & v = \int \sin 3x \, dx = -\frac{1}{3}\cos 3x \end{array} \right| \overset{(*)}{=}$$

$$\int \sin 3x \, dx = \left| \begin{array}{l} 3x = t \\ 3\,dx = dt \\ dx = \frac{1}{3}dt \end{array} \right| = \frac{1}{3}\int \sin t \, dt = -\frac{1}{3}\cos t + c = -\frac{1}{3}\cos 3x + c$$

$$\overset{(*)}{=} -\frac{1}{3}x^2 \cos 3x + \frac{2}{3}\int x \cos 3x \, dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{3} I_1$$

$$I_1 = \int x\cos 3x \, dx = \left| \begin{array}{ll} u = x & dv = \cos 3x \, dx \\ du = dx & v = \frac{1}{3}\sin 3x \end{array} \right| = \frac{1}{3}x \sin 3x - \frac{1}{3}\int \sin 3x \, dx$$

$$= \frac{1}{3}x\sin 3x - \frac{1}{3}\cdot\left(-\frac{1}{3}\right)\cos 3x + C_1 = \frac{1}{3}x \sin 3x + \frac{1}{9}\cos 3x + C_1$$

$$\int x^2 \sin 3x \, dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + C$$

3. $$\int x^3 \ln x \, dx = \left| \begin{array}{ll} u = \ln x & dv = x^3 dx \\ du = \frac{1}{x}dx & v = \frac{x^4}{4} \end{array} \right| = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{4}\cdot\frac{x^4}{4} + C = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

4. $$\int \arcsin x \, dx = \left| \begin{array}{ll} u = \arcsin x & dv = dx \\ du = \frac{1}{\sqrt{1-x^2}}dx & v = x \end{array} \right| = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x\arcsin x - I_1$$

$$I_1=\int\frac{x}{\sqrt{1-x^2}}\,dx=\left|\begin{array}{l}1-x^2=t^2\\ -2x\,dx=2t\,dt\\ x\,dx=-t\,dt\\ t=\sqrt{1-x^2}\end{array}\right|=\int\frac{-t\,dt}{t}=-\int dt=-t+C_1=-\sqrt{1-x^2}+C_1$$

$$\int\arcsin x\,dx=x\arcsin x+\sqrt{1-x^2}+C$$

5. ISPITNI ZADATAK

$$\int\sin(\ln x)\,dx=\left|\begin{array}{ll}u=\sin(\ln x) & dv=dx\\ du=\cos(\ln x)\cdot\frac{1}{x}\,dx & v=x\end{array}\right|=x\sin(\ln x)-\int\cos(\ln x)\,dx$$

$$\int\cos(\ln x)\,dx=\left|\begin{array}{ll}u=\cos(\ln x) & dv=dx\\ du=-\sin(\ln x)\cdot\frac{1}{x}\,dx & v=x\end{array}\right|=x\cos(\ln x)+\int\sin(\ln x)\,dx$$

$$\int\sin(\ln x)\,dx=x\sin(\ln x)-x\cos(\ln x)-\int\sin(\ln x)\,dx$$

$$2\int\sin(\ln x)\,dx=x\sin(\ln x)-x\cos(\ln x)\qquad /:2$$

$$\int\sin(\ln x)\,dx=\frac{x}{2}\sin(\ln x)-\frac{x}{2}\cos(\ln x)+C$$

6. $$\int x\,\mathrm{arctg}\,x\,dx=\left|\begin{array}{ll}u=\mathrm{arctg}\,x & dv=x\,dx\\ du=\frac{1}{1+x^2}\,dx & v=\frac{x^2}{2}\end{array}\right|=\frac{1}{2}x^2\,\mathrm{arctg}\,x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$\int\frac{x^2}{x^2+1}\,dx=\int\frac{x^2+1-1}{x^2+1}\,dx=\int dx-\int\frac{1}{x^2+1}\,dx=x-\mathrm{arctg}\,x+C_1$$

$$\int x\,\mathrm{arctg}\,x\,dx=\frac{1}{2}x^2\,\mathrm{arctg}\,x-\frac{1}{2}x+\frac{1}{2}\,\mathrm{arctg}\,x+C$$

7.^{V} $\int x^2e^{-2x}\,dx$ Rj: $-\frac{x^2}{2}e^{-2x}-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}+C$

8.^{V} $\int\frac{x^2}{(x^2+1)^2}\,dx$ Rj: $-\frac{x}{2(x^2+1)}+\frac{1}{2}\,\mathrm{arctg}\,x+C$

uputa: $\int x\cdot\frac{x}{(x^2+1)^2}\,dx=\left|\begin{array}{ll}u=x & dv=\frac{x\,dx}{(x^2+1)^2}\end{array}\right|=\ldots$

9. $$\int x^3\ln(2x+1)\,dx=\left|\begin{array}{ll}u=\ln(2x+1) & dv=x^3\,dx\\ du=\frac{1}{2x+1}\cdot 2\,dx & v=\frac{x^4}{4}\end{array}\right|=$$

$$=\frac{1}{4}x^4\ln(2x+1)-\frac{1}{4}\cdot 2\int\frac{x^4}{2x+1}\,dx=\frac{1}{4}x^4\ln(2x+1)-\frac{1}{2}I_1$$

$$x^4:(2x+1)=\frac{1}{2}x^3-\frac{1}{4}x^2+\frac{1}{8}x-\frac{1}{16}$$

$$\begin{array}{l}-\left(x^4+\frac{1}{2}x^3\right)\\ \hline -\frac{1}{2}x^3\\ -\left(-\frac{1}{2}x^3-\frac{1}{4}x^2\right)\\ \hline \frac{1}{4}x^2\\ -\left(\frac{1}{4}x^2+\frac{1}{8}x\right)\\ \hline -\frac{1}{8}x\\ -\left(-\frac{1}{8}x-\frac{1}{16}\right)\\ \hline \text{ostatak } \frac{1}{16}\end{array}$$

ostatak $\frac{1}{16}$

$$x^4=\left(\frac{1}{2}x^3-\frac{1}{4}x^2+\frac{1}{8}x-\frac{1}{16}\right)(2x+1)+\frac{1}{16}$$

$$I_1=\int\left(\frac{1}{2}x^3-\frac{1}{4}x^2+\frac{1}{8}x-\frac{1}{16}+\frac{\frac{1}{16}}{2x+1}\right)dx=$$

$$=\frac{1}{2}\cdot\frac{x^4}{4}-\frac{1}{4}\cdot\frac{x^3}{3}+\frac{1}{8}\cdot\frac{x^2}{2}-\frac{1}{16}x+\frac{1}{16}\cdot\frac{1}{2}\ln|2x+1|+C_1$$

$$\int x^3\ln(2x+1)\,dx=\frac{1}{4}x^4\ln(2x+1)-\frac{1}{16}x^4+\frac{1}{24}x^3-\frac{1}{32}x^2+\frac{1}{32}x-\frac{1}{64}\ln|2x+1|+C$$

10. ISPITNI ZADATAK

$$I=\int\sqrt{x}\ln^2x\,dx=\left|\begin{array}{ll}u=\ln^2x & dv=\sqrt{x}\,dx\\ du=2\ln x\cdot\frac{1}{x}\,dx & v=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2}{3}\sqrt{x^3}\end{array}\right|=$$

$$=\frac{2}{3}\sqrt{x^3}\ln^2x-2\cdot\frac{2}{3}\int\frac{\sqrt{x^3}}{x}\ln x\,dx=\frac{2}{3}x\sqrt{x}\ln^2x-\frac{4}{3}\int\sqrt{x}\ln x\,dx$$

$$\int\sqrt{x}\ln x\,dx=\left|\begin{array}{ll}u=\ln x & dv=\sqrt{x}\,dx\\ du=\frac{1}{x}\,dx & v=\frac{2}{3}\sqrt{x^3}\end{array}\right|=\frac{2}{3}x\sqrt{x}\ln x-\frac{2}{3}\int\frac{x\sqrt{x}}{x}\,dx=$$

$$=\frac{2}{3}x\sqrt{x}\ln x-\frac{2}{3}\cdot\frac{2}{3}\sqrt{x^3}+C=\frac{2}{3}x\sqrt{x}\ln x-\frac{4}{9}x\sqrt{x}+C$$

$$I=\frac{2}{3}x\sqrt{x}\ln^2x-\frac{8}{9}x\sqrt{x}\ln x+\frac{16}{27}x\sqrt{x}+C$$

11) ISPITNI ZADATAK

$$I=\int e^{3x}\cos 4x\,dx = \begin{vmatrix} u=e^{3x} & dv=\cos 4x\,dx \\ du=3e^{3x}dx & v=\frac14 \sin 4x\end{vmatrix} =$$

$$= \frac14 e^{3x}\sin 4x - \frac34\int e^{3x}\sin 4x\,dx = \frac14 e^{3x}\sin 4x - \frac34 I_1$$

$$I_1 = \int e^{3x}\sin 4x\,dx = \begin{vmatrix} u=e^{3x} & dv=\sin 4x\,dx \\ du=3e^{3x}dx & v=-\frac14 \cos 4x\end{vmatrix} =$$

$$= -\frac14 e^{3x}\cos 4x + \frac34\int e^{3x}\cos 4x\,dx = -\frac14 e^{3x}\cos 4x + \frac34 I$$

$$I = \frac14 e^{2x}\sin 4x + \frac{3}{16}e^{3x}\cos 4x - \frac{9}{16} I \qquad /\cdot 16$$

$$16 I = 4e^{2x}\sin 4x + 3e^{3x}\cos 4x - 9I$$

$$I = \frac{4e^{3x}\sin 4x+3e^{3x}\cos 4x}{25}+C$$

12) ISPITNI ZADATAK

$$I=\int x^2e^{3x}dx = \begin{vmatrix} u=x^2 & dv=e^{3x}dx \\ du=2x\,dx & v=\frac13 e^{3x}\end{vmatrix} = \frac13x^2e^{3x}-\frac23\int xe^{3x}dx$$

$$\int xe^{3x}dx = \begin{vmatrix} u=x & dv=e^{3x}dx \\ du=dx & v=\frac13 e^{3x}\end{vmatrix} = \frac13 xe^{3x}-\frac13\int e^{3x}dx = \frac13xe^{3x}-\frac19e^{3x}+C$$

$$I=\frac13x^2e^{3x}-\frac29xe^{3x}+\frac{2}{27}e^{3x}+C$$

13)ˇ $I=\int\frac{\arcsin\frac x2}{\sqrt{2-x}}dx$ Uputa: $u=\arcsin\frac x2$, $dv=\frac{dx}{\sqrt{2-x}}$ Rj: $-2\sqrt{2-x}\cdot\arcsin\frac x2 + 4\sqrt{2+x}+C$

14)ˇ $I=\int e^x\sin x\,dx$ Uputa: $u=e^x$ Rj: $\frac{e^x(\sin x-\cos x)}{2}+C$

15)ˇ $\int x^5\ln x\,dx$

17)ˇ $\int\frac{\ln(x^2+1)}{x^3}dx$

16)ˇ $\int\frac{x^2\,dx}{\cos^2x}$

18)ˇ $\int(\arcsin x)^2dx$

Izračunati integral $I=\int x^3e^{\frac x2}dx$.

Rj.

$$\int e^{\frac x2}dx = \begin{vmatrix} \frac x2=t \\ \frac12 dx=dt \\ dx=2dt\end{vmatrix} = 2\int e^t dt = 2e^t+C = 2e^{\frac x2}+C$$

$$\int x^3 e^{\frac x2}dx = \begin{vmatrix} u=x^3 & dv=e^{\frac x2}dx \\ du=3x^2dx & v=2e^{\frac x2}\end{vmatrix} = 2x^3e^{\frac x2}-6\int x^2e^{\frac x2}dx$$

$$\int x^2e^{\frac x2}dx = \begin{vmatrix} u=x^2 & dv=e^{\frac x2}dx \\ du=2x\,dx & v=2e^{\frac x2}\end{vmatrix} = 2x^2e^{\frac x2}-4\int xe^{\frac x2}dx$$

$$\int xe^{\frac x2}dx = \begin{vmatrix} u=x & dv=e^{\frac x2}dx \\ du=dx & v=2e^{\frac x2}\end{vmatrix} = 2xe^{\frac x2}-2\int e^{\frac x2}dx = 2xe^{\frac x2}-4e^{\frac x2}+C$$

$$I=2x^3e^{\frac x2}-6\left[2x^2e^{\frac x2}-4\left(2xe^{\frac x2}-4e^{\frac x2}\right)\right]+C=$$

$$=2x^3e^{\frac x2}-6\left(2x^2e^{\frac x2}-8xe^{\frac x2}+16e^{\frac x2}\right)+C$$

$$=2x^3e^{\frac x2}-12x^2e^{\frac x2}+48xe^{\frac x2}-96e^{\frac x2}+C$$

$$I=2e^{\frac x2}\left(x^3-6x^2+24x-48\right)+C$$

Izračunati: $\int \sqrt{5-x^2}\,dx$.

Rj. $\int \sqrt{5-x^2}\,dx = \left| \begin{array}{l} x=\sqrt{5}\,t \\ dx=\sqrt{5}\,dt \\ t=x/\sqrt{5} \end{array} \right| = \int \sqrt{5-5t^2}\,\sqrt{5}\,dt = 5\int \sqrt{1-t^2}\,dt$

$$\int \sqrt{1-t^2}\,dt = \left| \begin{array}{ll} u=\sqrt{1-t^2} & dv=dt \\ du=\frac{-2t}{2\sqrt{1-t^2}}=\frac{-t}{\sqrt{1-t^2}}dt & v=t \end{array} \right| = t\sqrt{1-t^2} + \int \frac{t^2}{\sqrt{1-t^2}}dt$$

$$\int \frac{t^2}{\sqrt{1-t^2}} = \int \frac{t^2-1+1}{\sqrt{1-t^2}}dt = -\int \frac{1-t^2}{\sqrt{1-t^2}}dt + \int \frac{dt}{\sqrt{1-t^2}} =$$

$$= -\int \sqrt{1-t^2}\,dt + \arcsin t$$

$$\int \sqrt{1-t^2}\,dt = t\sqrt{1-t^2} - \int \sqrt{1-t^2} + \arcsin t$$

$$2\int \sqrt{1-t^2}\,dt = t\sqrt{1-t^2} + \arcsin t$$

$$\int \sqrt{1-t^2} = \frac{1}{2}\left(t\sqrt{1-t^2} + \arcsin t\right) + C$$

$$\int \sqrt{5-x^2}\,dx = \frac{5}{2}\left(\frac{x}{\sqrt{5}}\sqrt{1-\frac{x^2}{5}} + \arcsin\frac{x}{\sqrt{5}}\right) + C =$$

$$= \frac{5}{2}\left(\frac{x\sqrt{5}}{5}\sqrt{\frac{5-x^2}{5}} + \arcsin\frac{x\sqrt{5}}{5}\right) + C =$$

$$= \frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\arcsin\frac{x\sqrt{5}}{5} + C$$

Izračunati integral $\int x\sqrt{1-x^4}\,dx$

Rj. $\int x\sqrt{1-x^4}\,dx = \int x\sqrt{(1-x^2)(1+x^2)}\,dx = \left| \begin{array}{l} x^2=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \end{array} \right| = \frac{1}{2}\int \sqrt{(1-t)(1+t)}\,dt =$

$$= \frac{1}{2}\int \sqrt{1-t^2}\,dt = \frac{1}{2}\int \frac{1-t^2}{\sqrt{1-t^2}}dt = \frac{1}{2}\left[\int \frac{dt}{\sqrt{1-t^2}} - \int \frac{t^2}{\sqrt{1-t^2}}dt\right]$$

$$\int \frac{t^2}{\sqrt{1-t^2}}dt = \left| \begin{array}{ll} u=t & dv=\frac{t}{\sqrt{1-t^2}}dt \\ du=dt & v=\int \frac{t}{\sqrt{1-t^2}}dt = \left| \begin{array}{l} 1-t^2=s^2 \\ -2t\,dt=2s\,ds \\ t\,dt=-s\,ds \end{array} \right| = -\int \frac{s\,ds}{s} = -\int ds \\ & = -s = -\sqrt{1-t^2} \end{array} \right| =$$

$$= -t\sqrt{1-t^2} + \int \sqrt{1-t^2}\,dt \qquad \text{Sad imamo:}$$

$$\underline{\underline{\frac{1}{2}\int \sqrt{1-t^2}\,dt}} = \frac{1}{2}\arcsin t + \frac{1}{2}t\sqrt{1-t^2} \underline{\underline{- \frac{1}{2}\int \sqrt{1-t^2}\,dt}}$$

$$\int \sqrt{1-t^2}\,dt = \frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t$$

vratimo smjene $\qquad \frac{1}{2}\int \sqrt{1-t^2} = \frac{1}{2}\left(\frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t\right)$

$$\int x\sqrt{1-x^4}\,dx = \frac{1}{4}x^2\sqrt{1-x^4} + \frac{1}{4}\arcsin x^2 + C$$

5. Integracija kvadratnog trinoma

U ovoj lekciji rješavamo integrale sljedećih oblika

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx\,; \quad \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx\,; \quad \int \sqrt{ax^2+bx+c}\,dx\,.$$

Kvadratni trinom ax^2+bx+c, koji se pojavljuje u podintegralnoj f-ji uvijek svodimo na "pogodniji" oblik:

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) =$$

$$= a\left(x^2+2\cdot x\cdot\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 + \underbrace{\frac{c}{a}-\frac{b^2}{4a^2}}_{\in\mathbb{R}}\right] =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 \pm d^2\right].$$

Odrediti integrale

a) $\displaystyle\int \frac{dx}{x^2+4x+8}$; b) $\displaystyle\int \frac{7-8x}{2x^3-3x+1}\,dx$;

c) $\displaystyle\int \frac{3x-2}{x^2+6x+9}\,dx$; d) $\displaystyle\int \frac{6x^3-7x^2+3x-1}{2x-3x^2}\,dx$.

Rj. a) $\displaystyle\int \frac{dx}{x^2+4x+8}$ Pro primjetimo da je

$x^2+4x+8 = x^2+2\cdot x\cdot 2+4+4 = (x+2)^2+4$

i da je $d(x+2)=dx$. Prema tome

$$\int \frac{dx}{x^2+4x+8} = \int \frac{d(x+2)}{(x+2)^2+4} = \frac{1}{2}\,\mathrm{arctg}\,\frac{x+2}{2} + C$$

Prema formuli 8. $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\,\mathrm{arctg}\,\frac{u}{a} + C$ pri čemu $u=x+2$, $a=2$

b)

$$2x^3-3x+1 = 2\left(x^3-\frac{3}{2}x+\frac{1}{2}\right) = 2\left(x^2-2\cdot x\cdot\frac{3}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+\frac{1}{2}\right)$$

$$= 2\left[\left(x-\frac{3}{4}\right)^2+\frac{1}{2}-\frac{9}{16}\right] = 2\left[\left(x-\frac{3}{4}\right)^2-\frac{1}{16}\right]$$

$$\int \frac{7-8x}{2x^3-3x+1}\,dx = \int \frac{7-8x}{2\left[\left(x-\frac{3}{4}\right)^2-\frac{1}{16}\right]}\,dx =$$

uvodimo smjenu $x-\frac{3}{4}=t$, $dx=dt$, $x=t+\frac{3}{4}$, $7-8x = 7-8t-6 = 1-8t$

$$= \frac{1}{2}\int \frac{1-8t}{t^2-\frac{1}{16}}\,dt = \frac{1}{2}\int \frac{dt}{t^2-\frac{1}{16}} - 4\int \frac{t\,dt}{t^2-\frac{1}{16}} =$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{1}{4}}\ln\left|\frac{t-\frac{1}{4}}{t+\frac{1}{4}}\right| - 4\int \frac{\frac{1}{2}\,d\left(t^2-\frac{1}{16}\right)}{t^2-\frac{1}{16}} =$$

$$= \ln\left|\frac{t-\frac{1}{4}}{t+\frac{1}{4}}\right| - 2\ln\left|t^2-\frac{1}{16}\right| + C =$$

$$= \ln\left|\frac{x-\frac{3}{4}-\frac{1}{4}}{t-\frac{3}{4}+\frac{1}{4}}\right| - 2\ln\left|\left(x-\frac{3}{4}\right)^2-\frac{1}{16}\right| + C$$

$$= \ln\left|\frac{x-1}{x-\frac{1}{2}}\right| - 2\ln\left|x^2-\frac{3}{2}x+\frac{1}{2}\right| + C$$

c) $\int \frac{3x-2}{x^2+6x+9}\,dx$ $\qquad x^2+6x+9 = x^2+2\cdot x\cdot 3+3^2 = (x+3)^2$

$$\int\frac{3x-2}{x^2+6x+9}\,dx = \int\frac{3x-2}{(x+3)^2}\,dx = \left|\begin{array}{l}\text{uvodimo smjenu}\\ x+3=t\\ dx=dt\\ x=t-3\\ 3x-2=3t-9-2=3t-11\end{array}\right| =$$

$$= \int\frac{3t-11}{t^2}\,dt = \int\left(\frac{3}{t}-\frac{11}{t^2}\right)dt = 3\int\frac{dt}{t} - 11\int t^{-2}\,dt =$$

$$= 3\ln|t| + 11t^{-1} + C = 3\ln|x+3| + \frac{11}{x+3} + C.$$

d) $J = \int\frac{6x^3-7x^2+3x-1}{2x-3x^2}\,dx$

Prvo podijelimo $6x^3-7x^2+3x-1$ sa $2x-3x^2$:

$$(6x^3-7x^2+3x-1):(-3x^2+2x) = -2x+1$$
$$-\underline{\;6x^3-4x^2\;}$$
$$= \quad -3x^2+3x-1$$
$$-\underline{\;-3x^2+2x\;}$$
$$= \quad x-1$$

Prema tome

$$\frac{6x^3-7x^2+3x-1}{2x-3x^2} = -2x+1+\frac{x-1}{2x-3x^2}$$

$$\int\frac{6x^3-7x^2+3x-1}{2x-3x^2}\,dx = \int\left(-2x+1+\frac{x-1}{2x-3x^2}\right)dx =$$

$$= -2\int x\,dx + \int dx + \int\frac{(x-1)\,dx}{2x-3x^2} = -x^2+x+J_1$$

$$J_1 = -\frac{1}{3}\int\frac{(x-1)\,dx}{x^2-\frac{2}{3}x} = -\frac{1}{3}\int\frac{x}{x\left(x-\frac{2}{3}\right)}\,dx + \frac{1}{3}\int\frac{dx}{x^2-\frac{2}{3}x}$$

$$= \left|\begin{array}{l} x^2-\frac{2}{3}x = x^2-2\cdot x\cdot\frac{2}{6}+\left(\frac{2}{6}\right)^2-\left(\frac{2}{6}\right)^2 \\ \quad = \left(x-\frac{1}{3}\right)^2-\frac{1}{9}\end{array}\right| =$$

$$= -\frac{1}{3}\int\frac{d\left(x-\frac{2}{3}\right)}{x-\frac{2}{3}} + \frac{1}{3}\int\frac{d\left(x-\frac{1}{3}\right)}{\left(x-\frac{1}{3}\right)^2-\frac{1}{9}} =$$

$$= -\frac{1}{3}\ln\left|x-\frac{2}{3}\right| + \frac{1}{3}\cdot\frac{1}{2\cdot\frac{1}{3}}\ln\left|\frac{x-\frac{1}{3}-\frac{1}{3}}{x-\frac{1}{3}+\frac{1}{3}}\right| =$$

$$= -\frac{1}{3}\ln\left|x-\frac{2}{3}\right| + \underbrace{\frac{1}{2}\ln\left|\frac{x-\frac{2}{3}}{x}\right|}_{=\frac{1}{2}\ln\left|x-\frac{2}{3}\right|-\frac{1}{2}\ln|x|}$$

$$J = C - x^2 + x + \frac{1}{6}\ln\left|x-\frac{2}{3}\right| - \frac{1}{2}\ln|x|$$

traženo rješenje

(#) Odrediti integrale

a) $\displaystyle\int \frac{dx}{\sqrt{x^3-4x-3}}$ b) $\displaystyle\int \frac{(3x-5)\,dx}{\sqrt{9+6x-3x^2}}$

Rj.

a) $x^2-4x-3 = x^2-2\cdot x\cdot 2+2^2-2^2-3 = (x-2)^2-7$

$$\int \frac{dx}{\sqrt{x^3-4x-3}} = \int \frac{dx}{\sqrt{(x-2)^2-7}} = \int \frac{d(x-2)}{\sqrt{(x-2)^2-7}} =$$

$$= \ln\left|x-2+\sqrt{(x-2)^2-7}\right| + C$$

Prema formuli 11. $\displaystyle\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right|+C$ pri čemu je $u=x-2$, $a=-7$

b)

$$9+6x-3x^2 = (-3)\left[x^2-2x-3\right] = (-3)\left(x^2-2\cdot x\cdot 1+1^2-1^2-3\right) =$$

$$=(-3)\left[(x-1)^2-4\right] = 3\left[4-(x-1)^2\right]$$

$$I=\int \frac{(3x-5)\,dx}{\sqrt{9+6x-3x^2}} = \int \frac{(3x-5)\,dx}{\sqrt{3\left[4-(x-1)^2\right]}} = \left| \begin{array}{ll} \text{uvodimo zamjenu} & \\ z=x-1 & \\ dz=dx & 3x-5= \\ x=z+1 & =3z+3-5 \\ & =3z-2 \end{array} \right| =$$

$$=\frac{1}{\sqrt{3}}\int \frac{3z-2}{\sqrt{4-z^2}}\,dz = \underbrace{\frac{3}{\sqrt{3}}}_{=\sqrt{3}} \underbrace{\int \frac{z\,dz}{\sqrt{4-z^2}}}_{=I_1} - \frac{2}{\sqrt{3}} \underbrace{\int \frac{dz}{\sqrt{4-z^2}}}_{=I_2}$$

$$I_1=\int \frac{z\,dz}{\sqrt{4-z^2}} = \left| \begin{array}{l} d(4-z^2)=-2z\,dz \\ z\,dz=-\frac{1}{2}\,d(4-z^2) \end{array} \right| = -\frac{1}{2}\int (4-z^2)^{-\frac{1}{2}}\,d(4-z^2) =$$

$$= -(4-z^2)^{\frac{1}{2}} + C_1$$

$$I_2 = \int \frac{dz}{\sqrt{4-z^2}} = \arcsin\frac{z}{2} + C_2$$

$$I = \sqrt{3}\left(-(4-z^2)^{\frac{1}{2}}\right) - \frac{2}{\sqrt{3}}\arcsin\frac{z}{2} + C =$$

$$= C - \sqrt{3(4-z^2)} - \frac{2}{\sqrt{3}}\arcsin\frac{z}{2} = \left| z=x-1 \right|$$

$$= C - \sqrt{9+6x-3x^2} - \frac{2}{\sqrt{3}}\arcsin\frac{x-1}{2}$$

\# Pomoću formule za parcijalnu integraciju $\int u\,dv = uv - \int v\,du$ odrediti integrale

a) $\int\sqrt{t^2+b}\,dt$; b) $\int\sqrt{a^2-t^2}\,dt$

pa uz pomoć dobijenog rezultata izračunati integrale

(i) $\int\sqrt{x^2-3}\,dx$

(ii) $\int\sqrt{x^2+2x+6}\,dx$

(iii) $\int\sqrt{3+4x-x^2}\,dx$

Rj.

a) $$I=\int\sqrt{t^2+b}\,dt = \left| \begin{matrix} u=\sqrt{t^2+b} & dv=dt \\ du=\dfrac{t\,dt}{\sqrt{t^2+b}} & v=t\end{matrix}\right| =$$

$$= t\sqrt{t^2+b} - \int \frac{t^2+b-b}{\sqrt{t^2+b}}\,dt = t\sqrt{t^2+b} - \int\frac{t^2+b}{\sqrt{t^2+b}}\,dt + b\int\frac{dt}{\sqrt{t^2+b}}$$

$$= t\sqrt{t^2+b} - \underbrace{\int\sqrt{t^2+b}\,dt}_{=I} + b\int\frac{dt}{\sqrt{t^2+b}} \Rightarrow$$

$$\Rightarrow 2I = t\sqrt{t^2+b} + b\int\frac{dt}{\sqrt{t^2+b}}$$

$$I = \frac12 t\sqrt{t^2+b} + \frac12 b\ln\left|t+\sqrt{t^2+b}\right|+C$$ traženo rješenje ...(*)

b) $$J=\int\sqrt{a^2-t^2}\,dt = \left| \begin{matrix} u=\sqrt{a^2-t^2} & dv=dt \\ du=\dfrac{-t}{\sqrt{a^2-t^2}}\,dt & v=t\end{matrix}\right| =$$

$$= t\sqrt{a^2-t^2} - \int\frac{-t^2+a^2-a^2}{\sqrt{a^2-t^2}}\,dt =$$

$$= t\sqrt{a^2-t^2} - \int\frac{a^2-t^2}{\sqrt{a^2-t^2}} + a^2\int\frac{dt}{\sqrt{a^2-t^2}}$$

$$= t\sqrt{a^2-t^2} - \underbrace{\int\sqrt{a^2-t^2}}_{=J} + a^2\int\frac{dt}{\sqrt{a^2-t^2}}$$

$$2J = t\sqrt{a^2-t^2} + a^2\int\frac{dt}{\sqrt{a^2-t^2}}$$

$$J = \int\sqrt{a^2-t^2}\,dt = \frac12 t\sqrt{a^2-t^2}+\frac12 a^2\arcsin\frac{t}{a}+C$$ ...(**)

Izračunajmo sad date integrale

(i) $$\int\sqrt{x^2-3}\,dx = \left|\begin{matrix}\text{po formuli (*)}\\ \text{gdje je}\\ t=x,\ b=-3\end{matrix}\right| = \frac{x}{2}\sqrt{x^2-3} - \frac32\ln\left|x+\sqrt{a^2-3}\right|+C$$

(ii) $$x^2+2x+6 = x^2+2\cdot x\cdot 1+1^2-1^2+6=(x+1)^2+5$$

$$\int\sqrt{x^2+2x+6}\,dx = \int\sqrt{(x+1)^2+5}\,d(x+1) = \left|\begin{matrix}\text{po formuli (*)}\\ \text{gdje je}\\ t=x+1,\ b=5\end{matrix}\right| =$$

$$= \frac{x+1}{2}\sqrt{(x+1)^2+5}+\frac52\ln\left|x+1+\sqrt{(x+1)^2+5}\right|+C$$

(iii) $$3+4x-x^2=(-1)(x^2-4x-3)=(-1)(x^2-2\cdot x\cdot 2+2^2-2^2-3) = -\left[(x-2)^2-7\right]=7-(x-2)^2$$

$$\int\sqrt{3+4x-x^2}\,dx = \int\sqrt{7-(x-2)^2}\,d(x-2) = \left|\begin{matrix}\text{po formuli (**)}\\ \text{gdje je } t=x-2,\ a^2=7\end{matrix}\right| =$$

$$= \frac{x-2}{2}\sqrt{7-(x-2)^2}+\frac72\arcsin\frac{x-2}{\sqrt{7}}+C.$$

Zadaci za vježbu

Odrediti integrale

1. $\int \frac{dx}{x^2-x-6}$ 2. $\int \frac{dx}{x^2+4x+29}$ 3. $\int \frac{dx}{4x-1-4x^2}$

4. $\int \frac{(4x-3)dx}{x^2+3x+4}$ 5. $\int \frac{(3x+4)dx}{x^2+5x}$ 6. $\int \frac{18x^2+13x}{1+6x+9x^2}dx$

7. $\int \frac{x^3-2x^2+4}{x^2+2x-3}dx$ 8. $\int \frac{dx}{\sqrt{2+x-x^2}}$ 9. $\int \frac{dx}{\sqrt{x^2-2x}}$

10. $\int \frac{(x+3)dx}{\sqrt{1-4x^2}}$ 11. $\int \frac{(x-3)dx}{\sqrt{x^2+6x}}$ 12.* $\int \frac{x\,dx}{\sqrt{1-2x-3x^2}}$

13. $\int \sqrt{x^2+4x}\,dx$ 14. $\int \sqrt{1-2x-x^2}\,dx$

Rješenja:

1. $\frac{1}{5}\,\mathrm{arctg}\,\frac{x+2}{5}$ 2. ? 3. $\frac{1}{4x-2}$ 4. $2\ln(x^2+3x+4)-\frac{18}{\sqrt{7}}\,\mathrm{arctg}\,\frac{2x+3}{\sqrt{7}}$

5. $\frac{4}{5}\ln|x|+\frac{11}{5}\ln|x+5|$ 6. $2x+\frac{1}{9}\left(\frac{7}{3x+1}+\ln|3x+1|\right)$ 7. $\frac{1}{2}x^2-4x+$

$+\frac{3}{4}\ln|x-1|+\frac{41}{4}\ln|x+3|$ 8. $\arcsin\frac{2x-1}{3}$ 9. $\ln\left|x-1+\sqrt{x^2-2x}\right|$

10. $\frac{3}{2}\arcsin 2x-\frac{1}{4}\sqrt{1-4x^2}$ 11. $\sqrt{x^2+6x}-6\ln\left|x+3+\sqrt{x^2+6x}\right|$

12. $-\frac{1}{3\sqrt{3}}\arcsin\frac{3x+1}{2}-\frac{1}{3}\sqrt{1-2x-3x^2}$ 13. $\frac{x+2}{2}\sqrt{x^2+4x}-$

$-2\ln\left|x+2+\sqrt{x^2+4x}\right|$ 14. $\frac{x+1}{2}\sqrt{1-2x-x^2}+\arcsin\frac{x+1}{\sqrt{2}}$.

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija kvadratnog trinoma)

$\int \frac{dx}{ax^2+b}$, $\int \frac{dx}{\sqrt{ax^2+b}}$, smjena $\sqrt{|a|}\cdot x=\sqrt{|b|}\cdot t$

1. $$\int \frac{dx}{4x^2+9}=\int\frac{dx}{(2x)^2+3^2}=\left|\begin{aligned}2x&=3t\\2dx&=3dt\\dx&=\tfrac{3}{2}dt\\t&=\tfrac{2x}{3}\end{aligned}\right|=\frac{3}{2}\int\frac{dt}{(3t)^2+3^2}=\frac{3}{2}\int\frac{dt}{9t^2+9}=$$

$$=\frac{3}{2}\cdot\frac{1}{9}\int\frac{dt}{t^2+1}=\frac{1}{6}\,\mathrm{arctg}\,t+C=\frac{1}{6}\,\mathrm{arctg}\,\frac{2x}{3}+C$$

2. $$\int\frac{dx}{\sqrt{2x^2+25}}=\int\frac{dx}{\sqrt{(\sqrt{2}x)^2+5^2}}=\left|\begin{aligned}\sqrt{2}x&=5t\\\sqrt{2}dx&=5dt\\dx&=\tfrac{5}{\sqrt{2}}dt\\t&=\tfrac{\sqrt{2}}{5}x\end{aligned}\right|=\frac{5}{\sqrt{2}}\int\frac{dt}{\sqrt{\underbrace{25t^2+25}_{25\cdot(t^2+1)}}}=$$

$$=\frac{5}{\sqrt{2}}\cdot\frac{1}{5}\int\frac{dt}{\sqrt{t^2+1}}=\frac{1}{\sqrt{2}}\cdot\ln\left|t+\sqrt{t^2+1}\right|+C=\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}}{5}x+\sqrt{\frac{2}{25}x^2+1}\right|+C$$

3. $$\int\frac{dx}{5x^2-49}=\int\frac{dx}{(\sqrt{5}x)^2-7^2}=\left|\begin{aligned}\sqrt{5}x&=7t\\\sqrt{5}dx&=7dt\\dx&=\tfrac{7}{\sqrt{5}}dt\\t&=\tfrac{\sqrt{5}x}{7}\end{aligned}\right|=\frac{7}{\sqrt{5}}\int\frac{dt}{49t^2-49}=\frac{7}{\sqrt{5}}\cdot\frac{1}{49}\int\frac{dt}{t^2-1}$$

$$=\frac{1}{7\sqrt{5}}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C=\frac{1}{14\sqrt{5}}\ln\left|\frac{\frac{\sqrt{5}}{7}x-1}{\frac{\sqrt{5}}{7}x+1}\right|+C=\frac{1}{14\sqrt{5}}\ln\left|\frac{\sqrt{5}x-7}{\sqrt{5}x+7}\right|$$

4. $$\int\frac{dx}{\sqrt{7-9x^2}}=\int\frac{dx}{\sqrt{(\sqrt{7})^2-(3x)^2}}=\left|\begin{aligned}3x&=\sqrt{7}t\\3dx&=\sqrt{7}dt\\dx&=\tfrac{\sqrt{7}}{3}dt\\t&=\tfrac{3x}{\sqrt{7}}\end{aligned}\right|=\frac{\sqrt{7}}{3}\int\frac{dt}{\sqrt{7-7t^2}}=\frac{\sqrt{7}}{3}\cdot\frac{1}{\sqrt{7}}\int\frac{dt}{\sqrt{1-t^2}}$$

$$=\frac{1}{3}\arcsin t+C=\frac{1}{3}\arcsin\left(\frac{3x}{\sqrt{7}}\right)+C$$

5.v $\int\frac{dx}{4x^2+11}$, Rj. $\frac{\sqrt{11}}{22}\,\mathrm{arctg}\,\frac{2\sqrt{11}x}{11}+C$

6.v $\int\frac{dx}{\sqrt{9x^2-16}}$, Rj. $\frac{1}{3}\ln\left|3x+\sqrt{9x^2-16}\right|+C$

$$\int \frac{dx}{ax^2+bx+c}, \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}, \qquad ax^2+bx+c = a(x-\alpha)^2+\beta$$

7. $I=\int \frac{dx}{x^2+6x+13}, \qquad x^2+6x+13 = x^2+2\cdot x\cdot 3+3^2+4 = (x+3)^2+4$

$$I=\int \frac{dx}{(x+3)^2+2^2} = \left| \begin{array}{l} x+3=2t \\ dx=2dt \\ t=\frac{x+3}{2} \end{array} \right| = 2\int \frac{dt}{4t^2+4} = 2\cdot\frac{1}{4}\int\frac{dt}{t^2+1} = \frac{1}{2}\operatorname{arc\,tg} t + C$$

$$= \frac{1}{2}\operatorname{arctg}\frac{x+3}{2} + C$$

8. $I=\int \frac{dx}{\sqrt{2-x-x^2}}, \qquad 2-x-x^2 = -x^2-x+2 = (-1)\left[x^2+x-2\right] =$

$$= (-1)\left(x^2+2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-2\right) =$$

$$= (-1)\left[\left(x+\frac{1}{2}\right)^2-\frac{9}{4}\right] = \frac{9}{4}-\left(x+\frac{1}{2}\right)^2$$

$$I=\int\frac{dx}{\sqrt{\frac{9}{4}-(x+\frac{1}{2})^2}} = \int\frac{dx}{\sqrt{(\frac{3}{2})^2-(x+\frac{1}{2})^2}} = \left| \begin{array}{l} x+\frac{1}{2}=\frac{3}{2}t \\ dx=\frac{3}{2}dt \\ t=\frac{2x+1}{3} \end{array} \right| = \frac{3}{2}\int\frac{dt}{\sqrt{\frac{9}{4}-\frac{9}{4}t^2}} =$$

$$= \frac{3}{2}\cdot\frac{1}{\frac{3}{2}}\int\frac{dt}{\sqrt{1-t^2}} = \frac{3}{2}\cdot\frac{2}{3}\cdot\arcsin t + C = \arcsin\frac{2x+1}{3} + C$$

9. $I=\int\frac{dx}{2x^2-7x+3}, \qquad 2x^2-7x+3 = 2\cdot\left(x^2-\frac{7}{2}x+\frac{3}{2}\right) = 2\cdot\left(x^2-2\cdot x\cdot\frac{7}{4}+\left(\frac{7}{4}\right)^2-\right.$

$$\left.-\left(\frac{7}{4}\right)^2+\frac{3}{2}\right) = 2\cdot\left[\left(x-\frac{7}{2}\right)^2+\frac{-49+24}{16}\right] = 2\left[\left(x-\frac{7}{2}\right)^2-\frac{25}{16}\right]$$

$$I=\frac{1}{2}\int\frac{dx}{(x-\frac{7}{2})^2-\frac{25}{16}} = \frac{1}{2}\int\frac{dx}{(x-\frac{7}{2})^2-(\frac{5}{4})^2} = \left| \begin{array}{l} x-\frac{7}{4}=\frac{5}{4}t \\ dx=\frac{5}{4}dt \\ t=\frac{4x-14}{5} \end{array} \right| = \frac{1}{2}\cdot\frac{5}{4}\int\frac{dt}{\frac{25}{16}t^2-\frac{25}{16}}$$

$$= \frac{\cancel{5}^{1}}{\cancel{8}_{1}}\cdot\frac{\cancel{16}^{2}}{\cancel{25}_{5}}\int\frac{dt}{t^2-1} = \frac{2}{5}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C = \frac{1}{5}\ln\left|\frac{\frac{4x-12}{5}-1}{\frac{4x-2}{5}+1}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{\frac{4x-12}{5}}{\frac{4x-2}{5}}\right| + C = \frac{1}{5}\ln\left|\frac{4x-12}{4x-2}\right| + C = \frac{1}{5}\ln\left|\frac{2x-6}{2x-1}\right| + C$$

10. $\checkmark$ $\int\frac{dx}{\sqrt{x^2+8x+25}}, \qquad R_j:\ \ln\left|\frac{x+4}{3}+\sqrt{\left(\frac{x+4}{3}\right)^2+1}\right| + C$

$$\int\frac{mx+n}{ax^2+bx+c}\,dx, \quad \int\frac{mx+n}{\sqrt{ax^2+bx+c}}\,dx$$

11. $I=\int\frac{x+4}{x^2-4x+5}\,dx$

$(x^2-4x+5)' = 2x-4$

$x+4 = a\cdot(2x-4)+b, \quad a,b=?$

$x+4 = 2ax-4a+b$

$2a=1 \qquad -4a+b=4$

$a=\frac{1}{2} \qquad -2+b=4$

$b=6$

$$I=\int\frac{\frac{1}{2}(2x-4)+6}{x^2-4x+5}\,dx$$

$$= \underbrace{\frac{1}{2}\int\frac{2x-4}{x^2-4x+5}\,dx}_{I_1} + 6\underbrace{\int\frac{dx}{x^2-4x+5}}_{I_2} = \frac{1}{2}I_1+6I_2 \qquad x^2-2\cdot x\cdot 2+2^2-2^2+5 = (x-2)^2+1$$

$$I_1 = \left| \begin{array}{l} x^2-4x+5=t \\ (2x-4)dx=dt \end{array} \right| = \int\frac{dt}{t} = \ln|t|+C_1 = \ln|x^2-4x+5|+C_1$$

$$I_2 = \int\frac{dx}{(x-2)^2+1} = \left| \begin{array}{l} x-2=t \\ dx=dt \end{array} \right| = \int\frac{dt}{t^2+1} = \operatorname{arc\,tg} t + C_2 = \operatorname{arctg}(x-2)+C_2$$

$$I = \frac{1}{2}\ln|x^2-4x+5| + \operatorname{arctg}(x-2) + C$$

12. $I=\int\frac{x}{\sqrt{x^2+2x-5}}\,dx$

$(x^2+2x-5)' = 2x+2 \qquad 2a=1 \Rightarrow a=\frac{1}{2}$

$x=a(2x+2)+b \qquad 2a+b=0$

$x=2ax+2a+b \qquad 2\cdot\frac{1}{2}+b=0$

$b=-1$

$$I=\int\frac{\frac{1}{2}(2x+2)-1}{\sqrt{x^2+2x-5}}\,dx = \frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x-5}}\,dx - \int\frac{dx}{\sqrt{x^2+2x-5}} = \frac{1}{2}I_1 - I_2$$

$$\int\frac{2x+2}{\sqrt{x^2+2x-5}}\,dx = \left| \begin{array}{l} x^2-2x-5=t \\ (2x-2)dx=dt \end{array} \right| = \int\frac{dx}{\sqrt{t}} = \frac{1}{2}\sqrt{t}+C_1 = \frac{1}{2}\sqrt{x^2+2x-5}+C_1$$

$x^2+2x-5 = x^2+2\cdot x\cdot 1+1^2-1^2-5 =$

$= (x+1)^2-6$

$$I_2=\int\frac{dx}{\sqrt{(x+1)^2-6}} = \left| \begin{array}{l} x+1=\sqrt{6}\,t \\ dx=\sqrt{6}\,dt \\ t=\frac{x+1}{\sqrt{6}} \end{array} \right| =$$

$$= \sqrt{6}\int \frac{dt}{\sqrt{6t^2-6}} = \frac{\sqrt{6}}{\sqrt{6}}\int\frac{dt}{\sqrt{t^2-1}} = \ln\left|t+\sqrt{t^2-1}\right|+C_2$$

$$= \ln\left|\frac{x+1}{\sqrt{6}}+\sqrt{\frac{(x+1)^2}{6}-1}\right|+C_2$$

$$I = \sqrt{x^2+2x-5} - \ln\left|\frac{x+1}{\sqrt{6}}+\sqrt{\frac{(x+1)^2}{6}-1}\right|+C$$

13. $I=\int \frac{6x-7}{x^2-4x+5}\,dx$, $\quad (x^2-4x+5)'=2x-4$, $\quad 6x-7=a(2x-4)+b$

$6x-7=2ax-4a+b$

$2a=6 \Rightarrow a=3$

$b-4a=-7 \Rightarrow b=5$

$x^2-4x+5 = x^2-2\cdot x\cdot 2+2^2+5-2^2=(x-2)^2+1$

$$I=\int\frac{3(2x-4)+5}{x^2-4x+5}\,dx = 3\int\frac{2x-4}{x^2-4x+5}\,dx+5\int\frac{dx}{x^2-4x+5}=3I_1+5I_2$$

$$\int \frac{2x-4}{x^2-4x+5}\,dx = \left|\begin{matrix}x^2-4x+5=t\\(2x-4)\,dx=dt\end{matrix}\right| = \int\frac{dt}{t}=\ln|t|+C_1=\ln|x^2-4x+5|+C_1$$

$$I_2=\int\frac{dx}{(x-2)^2+1} = \left|\begin{matrix}x-2=t\\dx=dt\end{matrix}\right| = \int \frac{dt}{t^2+1} = \text{arc tg}\, t + C_2 =$$
$$= \text{arc tg}(x-2)+C_2$$

$$I = 3\ln|x^2-4x+5| + 5\,\text{arc tg}(x-2)+C$$

14. $\int\frac{3x+2}{\sqrt{x^2-8x-9}}\,dx$, $\quad$ Rj: $3\sqrt{x^2-8x-9}+14\ln\left|\frac{x-4}{5}+\sqrt{\left(\frac{x-4}{5}\right)^2-1}\right|+C$

15. $\int\frac{3x+4}{\sqrt{-x^2+6x-8}}\,dx$, $\quad$ Rj: $-3\sqrt{-x^2+6x-8}+13\arcsin(x-3)$

16. $\int\frac{2x+3}{\sqrt{4x^2+4x+3}}\,dx$

17. $\int\frac{x-4}{x^2-5x+6}\,dx$

Dio tablice integrala

1. $\int u^a\,du=\frac{u^{a+1}}{a+1}+C,\ a\neq-1.$

2. $\int u^{-1}\,du=\int\frac{du}{u}=\int\frac{u'}{u}\,dx=$
$=\ln|u|+C.$

3. $\int a^u\,du=\frac{a^u}{\ln a}+C;\ \int e^u\,du=$
$=e^u+C.$

4. $\int \sin u\,du=-\cos u+C.$

5. $\int \cos u\,du=\sin u+C.$

6. $\int \sec^2 u\,du=\text{tg}\,u+C.$

7. $\int \text{cosec}^2 u\,du=-\text{ctg}\,u+C.$

8. $\int\frac{du}{u^2+a^2}=\frac{1}{a}\,\text{arc tg}\,\frac{u}{a}+C.$

9. $\int\frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$

10. $\int\frac{du}{\sqrt{a^2-u^2}}=\text{arc sin}\frac{u}{a}+C.$

11. $\int\frac{du}{\sqrt{u^2+a}}=$
$=\ln\left|u+\sqrt{u^2+a}\right|+C.$

6. Integracija trigonometriskih f-ja

Dosta često se pojavljuju integrali izraza, koji sadrže trigonometriske f-je sljedećih tipova:

I. $\int \sin^n x\,dx$, $\int \cos^n x\,dx$ II. $\int \sin^m x \cos^n x\,dx$

III. $\int tg^n x\,dx$, $\int ctg^n x\,dx$

gdje su m i n - pozitivni cijeli brojevi,

IV. $\int \sin ax \cos bx\,dx$, $\int \sin ax \sin bx\,dx$,

$\int \cos ax \cos bx\,dx$

koji se mogu svesti na formulu integriranja, a prematome i naći, tako što će se slijediti neko od pravila:

1. Integrali od parnog stepena sinusa ili kosinusa mogu se odrediti pomoću smanjivanja stepena (dvaput) pomoću formula:

$\sin^2 u = \frac{1}{2}(1-\cos 2u)$; $\cos^2 u = \frac{1}{2}(1+\cos 2u)$; $\sin u \cos u = \frac{1}{2}\sin 2u$.

2. Integrale neparnog stepena od sinusa ili kosinusa možemo odrediti putem razdvajanja jednog od drugog faktora i zamjenom komplementarne f-ju

novom promjenjivom.

3. Integrale tipa II možemo odrediti po pravilu 1, ako su oba broja m i n parna, ili po pravilu 2 ako su m ili n (ili oba) neparna.

4. Integrale tipa III možemo odrediti putem zamjene $tg\,x$, ili dosljedno, $ctg\,x$ novom promjenjivom.

5. Integrale tipa IV možemo odrediti tako što ćemo razložiti podintegralni izraz na dijelove pomoću formula

$$\sin ax \cos bx = \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]$$

$$\sin ax \sin bx = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]$$

$$\cos ax \cos bx = \frac{1}{2}[\cos(a+b)x + \cos(a-b)x]$$

(#) Odrediti integrale

a) $\int \sin^2 3x\,dx$; b) $\int \cos^4 x\,dx$; c) $\int \sin^5 x\,dx$;

d) $\int \sin^4 x \cos^2 x\,dx$; e) $\int \sin^6 kx \cos^3 kx\,dx$; f) $\int \sin^3 x \cos^5 x\,dx$.

Rj.

a) Prema pravilu 1 imamo

$$\int \sin^2 3x\,dx = \frac{1}{2}\int (1-\cos 6x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\cdot\frac{1}{6}\int \cos 6x\,d(6x) =$$

$$= \frac{x}{2} - \frac{1}{12}\sin 6x + C.$$

b) Prema pravilu 1 imamo

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2 dx = \frac{1}{4}\int (1+\cos 2x)^2 dx =$$

$$= \frac{1}{4}\left[\int dx + 2\int \cos 2x \cdot \frac{1}{2}\,d(2x) + \underbrace{\int \cos^2 2x\,dx}_{I_1}\right]$$

$$I_1 = \int \cos^2 2x\,dx = \frac{1}{2}\int (1+\cos 4x)\,dx = \frac{1}{2}\int dx + \frac{1}{2}\cdot\frac{1}{4}\int \cos 4x\,d(4x) =$$

$$= \frac{1}{2}x + \frac{1}{8}\sin 4x$$

Prema tome

$$\int \cos^4 x\,dx = \frac{1}{4}\left(x + \sin 2x + \frac{x}{2} + \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

c) $$\int \sin^5 x\,dx = \left| \begin{array}{l} \text{prema pravilu 2} \\ \sin^5 x = \sin^4 x \cdot \sin x \end{array} \right| = \int (\sin^2 x)^2 \sin x\,dx =$$

$$= \int (1-\cos^2 x)^2 \sin x\,dx = \left| \begin{array}{l} \cos x = z \\ -\sin x\,dx = dz \\ \sin x\,dx = -dz \end{array} \right| = \int (1-z^2)^2(-dz) =$$

$$= -\int (1-2z^2+z^4)\,dz = -z + \frac{2}{3}z^3 - \frac{1}{5}z^5 + C =$$

$$= C - \cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$$

d) $$\int \sin^4 x \cos^2 x\,dx = \left| \begin{array}{l} \text{prema pravilu 3} \\ \sin u \cos u = \frac{1}{2}\sin 2u \\ \sin^2 = \frac{1}{2}(1-\cos 2\ ) \end{array} \right| =$$

$$= \int \sin^2 x \sin^2 x \cos^2 x\,dx = \int \sin^2 x\,(\sin x \cos x)^2\,dx =$$

$$= \int \frac{1}{2}(1-\cos 2x)\cdot\frac{1}{4}\sin^2 2x\,dx = \frac{1}{8}\underbrace{\int \sin^2 2x\,dx}_{I_1} -$$

$$- \frac{1}{8}\underbrace{\int \sin^2 2x \cos 2x\,dx}_{I_2} = \frac{1}{8}\left(I_1 - I_2\right)$$

$$I_1 = \int \sin^2 2x\,dx = \frac{1}{2}\int (1-\cos 4x)\,dx = \frac{1}{2}\int dx - \frac{1}{8}\int \cos 4x\,d(4x)$$

$$= \frac{1}{2}x - \frac{1}{8}\sin 4x$$

$$I_2 = \int \sin^2 2x \cos 2x\,dx = \left| \begin{array}{l} \sin 2x = z \\ 2\cos 2x\,dx = dz \end{array} \right| = \frac{1}{2}\int z^2 dz = \frac{1}{2}\cdot\frac{z^3}{3}$$

$$= \frac{1}{6}\sin^3 2x.$$

Prema tome

$$\int \sin^4 x \cos^2 x \, dx = \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{1}{6}\sin^3 2x\right) + C$$

e) $$\int \sin^6 kx \cos^3 kx \, dx = \left| \begin{array}{l} \text{prema pravilu 3 pišemo} \\ \cos^3 kx = \cos^2 kx \cdot \cos kx \\ = (1-\sin^2 kx)\cos kx \end{array} \right| =$$

$$= \int \sin^6 kx \,(1-\sin^2 kx)\cos kx \, dx = \left| \begin{array}{l} \sin kx = z \\ k\cos kx \, dx = dz \end{array} \right| =$$

$$= \frac{1}{k}\int z^6 (1-z^2)\, dz = \frac{1}{k}\left(\int z^6 dz - \int z^8 dz\right) =$$

$$= \frac{1}{k}\left(\frac{z^7}{7} - \frac{z^9}{9}\right) + C = \frac{1}{7k}\sin^7 kx - \frac{1}{9k}\sin^9 kx + C$$

f) $$\int \sin^3 x \cos^5 x \, dx = \left| \begin{array}{l} \text{prema pravilu 3} \\ \sin^3 x = \sin^2 x \sin x \\ = (1-\cos^2 x)\sin x \end{array} \right| =$$

$$= \int (1-\cos^2 x)\cos^5 x \sin x \, dx = \left| \begin{array}{l} \cos x = z \\ -\sin x \, dx = dz \end{array} \right| =$$

$$= -\int (1-z^2) z^5 dz = -\int z^5 dz + \int z^7 dz = \frac{1}{8}z^8 - \frac{1}{6}z^6 + C$$

$$= \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + C.$$

Odrediti integrale

a) $\int \operatorname{tg}^4 x \, dx$ b) $\int \sin 3x \cos 5x \, dx$

Rj.

a) $$\int \operatorname{tg}^4 x \, dx = \left| \begin{array}{l} \text{primjenom pravila 4} \\ \operatorname{tg} x = z \\ x = \operatorname{arc\,tg} z \\ dx = \frac{dz}{1+z^2} \end{array} \right| = \int \frac{z^4}{z^2+1} dz \overset{(*)}{=}$$

$$\begin{array}{l} z^4 : (z^2+1) = z^2 - 1 \\ \underline{-(z^4 + z^2)} \\ \quad -z^2 \\ \underline{-(-z^2 - 1)} \\ \qquad 1 \end{array} \qquad \frac{z^4}{z^2+1} = z^2 - 1 + \frac{1}{z^2+1}$$

$$\overset{(*)}{=} \int \left(z^2 - 1 + \frac{1}{z^2+1}\right) dz = \frac{1}{3}z^3 - z + \operatorname{arc\,tg} z + C =$$

$$= \frac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x + C.$$

b) $$\int \sin 3x \cos 5x \, dx = \left| \begin{array}{l} \text{primjenom pravila 5} \\ \sin 3x \cos 5x = \frac{1}{2}[\sin 8x + \sin(-2x)] \end{array} \right| =$$

$$= \frac{1}{2}\int [\sin 8x - \sin 2x]\, dx = \frac{1}{2}\cdot\frac{1}{8}\int \sin 8x \, d(8x) -$$

$$- \frac{1}{2}\cdot\frac{1}{2}\int \sin 2x \, d(2x) = -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + C$$

Zadaci za vježbu

Odrediti integrale

1. $\int \cos^2 5x \, dx$ 2. $\int \cos^5 x \, dx$ 3. $\int \sin^2 x \cos^2 x \, dx$

4. $\int \sin^3 x \cos^2 x \, dx$ 5. $\int \sin^3 x \cos^3 x \, dx$ 6. $\int \sin^4 x \, dx$

7. $\int \mathrm{ctg}^4 y \, dy$ 8. $\int \cos \frac{4}{3}x \cos 3x \, dx$ 9. $\int \sin 5x \sin 6x \, dx$

10. $\int \sin at \cos bt \, dt$ 11.* $\int \sin 3x \sin 4x \sin 5x \, dx$

12.* $\int (\mathrm{tg}\, z + \mathrm{ctg}\, z)^3 \, dz$

Rješenja:

1. $\frac{x}{2} + \frac{1}{20}\sin 10x$ 2. $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x$ 3. $\frac{x}{8} - \frac{1}{32}\sin 4x$

4. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$ 5. $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x$ 6. $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$

7. $y + \mathrm{ctg}\, y - \frac{1}{3}\mathrm{ctg}^3 y$ 8. $\frac{3}{26}\sin\frac{13}{3}x + \frac{3}{10}\sin\frac{5}{3}x$

9. $\frac{1}{2}\sin x - \frac{1}{22}\sin 11x$ 10. $\frac{\cos(a-b)t}{2(b-a)} - \frac{\cos(a+b)t}{2(a+b)}$

11. $\frac{\cos 12x}{48} - \frac{\cos 6x}{24} - \frac{\cos 4x}{16} - \frac{\cos 2x}{8}$

12. $\frac{1}{2}(\mathrm{tg}^2 z - \mathrm{ctg}^2 z) + 2\ln|\mathrm{tg}\, z|$.

Izabrani **Zadaci za vježbu** sa rješenjima
(iz lekcije Integracija trigonometriskih funkcija)

$$\int \sin^m x \cdot \cos^n x \, dx \quad (m, n \in \mathbb{N}_0)$$

ako je m neparan uvodimo smjenu $\cos x = t$

ako je n neparan uvodimo smjenu $\sin x = t$

ako su m i n parni koristimo formule

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

1. $$\int \sin^3 x \cdot \cos^{12} x \, dx = \int \sin x \cdot \sin^2 x \cdot \cos^{12} x \, dx = \int \underline{\sin x}(1-\cos^2 x)\cos^{12} x \, \underline{dx}$$

$$= \left| \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \\ \sin x \, dx = -dt \end{array} \right| = \int (1-t^2) \cdot t^{12} (-dt) = -\int (t^{12} - t^{14}) \, dt = -\int t^{12} dt +$$

$$+ \int t^{14} dt = -\frac{t^{13}}{13} + \frac{t^{15}}{15} + C = -\frac{1}{13}\cos^{13} x + \frac{1}{15}\cos^{15} x + C$$

2. $$\int \sin^4 x \cdot \cos^5 x \, dx = \int \sin^4 x \cdot \cos^4 x \cdot \cos x \, dx = \int \sin^4 x (\cos^2 x)^2 \cos x \, dx$$

$$= \int \sin^4 x (1-\sin^2 x)^2 \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right| = \int t^4 \cdot (1-t^2)^2 \, dt =$$

$$= \int t^4 (1 - 2t^2 + t^4) \, dt = \int (t^8 - 2t^6 + t^4) \, dt = \frac{t^9}{9} - 2 \cdot \frac{t^7}{7} + \frac{t^5}{5} + C$$

$$= \frac{1}{9}\sin^9 x - \frac{2}{7}\sin^7 x + \frac{1}{5}\sin^5 x + C$$

3.$^\vee$ $\int \sin^7 x \cdot \cos^{10} x \, dx$ 5.$^\vee$ $\int \sin^5 x \, dx$

Rj. $-\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

4.$^\vee$ $\int \sin^2 x \cdot \cos^3 x \, dx$

6. $$\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx =$$

$$=\int \frac{1+2\cos 2x+\cos^2 2x}{4}\,dx = \frac{1}{4}\int dx + \frac{2}{4}\int \cos 2x\,dx + \frac{1}{4}\int \cos^2 2x\,dx =$$

$$=\frac{1}{4}x + \frac{1}{2}\cdot\frac{1}{2}\sin 2x + \frac{1}{4}\int \frac{1+\cos 4x}{2}\,dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\left(\int dx + \int \cos 4x\,dx\right)$$

$$\int \cos 2x\,dx = \left|\begin{matrix} 2x=t \\ 2dx=dt \\ dx=\frac{dt}{2} \end{matrix}\right| = \int \cos t\cdot\frac{dt}{2} = \frac{1}{2}\sin t + C = \frac{1}{2}\sin 2x + C$$

$$=\frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{4}\sin 4x + C = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Napomena: Zadatak možemo riješiti i parcijalnom integracijom $\int \cos^4 x\,dx = \int \cos^3 x\cdot\cos x\,dx = \left| \begin{matrix} u=\cos^3 x & dv=\cos x\,dx \\ \vdots & \end{matrix}\right|$

7. $$\int \sin^2 x\cdot\cos^2 x\,dx = \int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{4}\int(1-\cos^2 2x)\,dx$$

$$=\frac{1}{4}\int dx - \frac{1}{4}\int \cos^2 2x\,dx = \frac{1}{4}x - \frac{1}{4}\int \frac{1+\cos 4x}{2}\,dx = \frac{1}{4}x - \frac{1}{8}\int(1+\cos 4x)\,dx$$

$$=\frac{1}{4}x - \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x\,dx = \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\cdot\frac{1}{4}\sin 4x + C =$$

$$=\frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

8. $\int \sin^6 x\,dx$

9. $\int \sin^2 x\cdot\cos^4 x\,dx$

10. $\int \cos^6 x\,dx$

Rj. $\frac{5}{16}x - \frac{1}{48}\sin^3 2x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + C$

11. $\int \sin^4 2x\,dx$

Rj. $\frac{3}{8}x - \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x + C$

$$\int \sin\alpha x\cdot\sin\beta x\,dx,\quad \int \sin\alpha x\cdot\cos\beta x\,dx,\quad \int \cos\alpha x\cdot\cos\beta x\,dx$$

koristimo formule:

$$\sin\alpha x\cdot\sin\beta x = \frac{1}{2}\left[\cos(\alpha-\beta)x - \cos(\alpha+\beta)x\right]$$

$$\sin\alpha x\cdot\cos\beta x = \frac{1}{2}\left[\sin(\alpha+\beta)x + \sin(\alpha-\beta)x\right]$$

$$\cos\alpha x\cdot\cos\beta x = \frac{1}{2}\left[\cos(\alpha+\beta)x + \cos(\alpha-\beta)x\right]$$

12. $$\int \sin 4x\cdot\sin 2x\,dx = \frac{1}{2}\int(\cos 2x - \cos 6x)\,dx = \frac{1}{2}\int \cos 2x\,dx - \frac{1}{2}\int \cos 6x\,dx = \frac{1}{2}\cdot\frac{1}{2}\sin 2x - \frac{1}{2}\cdot\frac{1}{6}\sin 6x + C = \frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x + C$$

13. $$\int \sin 3x\cdot\cos 5x\,dx = \frac{1}{2}\int(\sin 8x + \sin(-2x))\,dx = \frac{1}{2}\int \sin 8x\,dx - \frac{1}{2}\int \sin 2x\,dx = \frac{1}{2}\cdot\left(-\frac{1}{8}\right)\cos 8x - \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cos 2x + C = -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + C$$

14. $$\int \cos x\cdot\cos 3x\cdot\cos 5x\,dx = \int \frac{1}{2}(\cos 4x + \cos(-2x))\cos 5x\,dx =$$

$$=\frac{1}{2}\int(\cos 4x + \cos 2x)\cos 5x\,dx = \frac{1}{2}\int \cos 4x\cos 5x\,dx + \frac{1}{2}\int \cos 2x\cos 5x\,dx$$

$$=\frac{1}{2}\cdot\frac{1}{2}\int(\cos 9x + \cos x)\,dx + \frac{1}{2}\cdot\frac{1}{2}\int(\cos 7x + \cos 3x)\,dx =$$

$$=\frac{1}{4}\int \cos 9x\,dx + \frac{1}{4}\int \cos x\,dx + \frac{1}{4}\int \cos 7x\,dx + \frac{1}{4}\int \cos 3x\,dx =$$

$$=\frac{1}{4}\cdot\frac{1}{9}\sin 9x + \frac{1}{4}\sin x + \frac{1}{4}\cdot\frac{1}{7}\sin 7x + \frac{1}{4}\cdot\frac{1}{3}\sin 3x + C$$

$$=\frac{1}{36}\sin 9x + \frac{1}{4}\sin x + \frac{1}{28}\sin 7x + \frac{1}{12}\sin 3x + C$$

15. $\int \sin x\cdot\sin 2x\cdot\sin 4x\,dx$ Rj. $-\frac{1}{20}\cos 5x - \frac{1}{12}\cos 3x + \frac{1}{28}\cos 7x + \frac{1}{4}\cos x + C$

16. $\int \sin 2x \cdot \cos 3x \cdot \sin 5x \, dx$

17. $\int \sin^2 \frac{x}{2} \cos 3x \, dx = \int \frac{1-\cos 2\cdot\frac{x}{2}}{2} \cos 3x \, dx = \frac{1}{2}\int (1-\cos x)\cdot \cos 3x \, dx$

$= \frac{1}{2}\int (\cos 3x - \cos x \cdot \cos 3x)\, dx = \frac{1}{2}\int \cos 3x \, dx - \frac{1}{2}\int \cos x \cos 3x \, dx$

$= \frac{1}{2}\cdot\frac{1}{3}\sin 3x - \frac{1}{2}\cdot\frac{1}{2}\int (\cos 4x + \cos 2x)\, dx = \frac{1}{6}\sin 3x - \frac{1}{4}\int \cos 4x \, dx$

$- \frac{1}{4}\int \cos 2x \, dx = \frac{1}{6}\sin 3x - \frac{1}{16}\sin 4x - \frac{1}{8}\sin 2x + C$

18. $\int \sin^2 2x \cdot \cos^2 3x \, dx$

19. $\int \sin^3 x \cdot \cos^2 2x \, dx$

\# Izračunati integral $I = \int \frac{8\cos x - \sin x}{2\cos x + \sin x}\, dx$,

R.j. $(2\cos x + \sin x)' = -2\sin x + \cos x$

$$\frac{8\cos x - \sin x}{2\cos x + \sin x} = A + B\,\frac{-2\sin x + \cos x}{2\cos x + \sin x} \quad / \cdot (2\cos x + \sin x)$$

$$8\cos x - \sin x = A(2\cos x + \sin x) + B(-2\sin x + \cos x)$$

$$8\cos x - \sin x = (2A + B)\cos x + (A - 2B)\sin x$$

$2A + B = 8$

$A - 2B = -1 \quad / \cdot 2$

$2A + B = 8$

$-\ 2A - 4B = -2$

$5B = 10$

$B = 2$

$2A + B = 8$

$2A = 8 - 2$

$2A = 6$

$A = 3$

$$I = \int \frac{8\cos x - \sin x}{2\cos x + \sin x}\, dx = \int \left(3 + 2\,\frac{-2\sin x + \cos x}{2\cos x + \sin x}\right) dx =$$

$$= 3\int dx + 2\int \frac{-2\sin x + \cos x}{2\cos x + \sin x}\, dx = \left| \begin{array}{l} 2\cos x + \sin x = t \\ (-2\sin x + \cos x)\,dx = dt \end{array} \right|$$

$$= 3x + 2\int \frac{dt}{t} = 3x + 2\ln|t| + C =$$

$$= 3x + 2\ln|2\cos x + \sin x| + C$$

7. Integracija racionalnih f-ja

Funkciju oblika $\frac{P(x)}{Q(x)}$, gdje su $P(x)$ i $Q(x)$ polinomi, nazivamo RACIONALNA F-JA. Ako je stepen polinoma $P(x)$ manji od stepena polinoma $Q(x)$, tada kažemo da je f-ja $\frac{P(x)}{Q(x)}$ PRAVA RACIONALNA F-JA. Ako je stepen polinoma $P(x)$ veći ili jednak od stepena polinoma $Q(x)$, tada je funkcija $\frac{P(x)}{Q(x)}$ NEPRAVA RACIONALNA F-JA.

Ako je racionalna f-ja $\frac{P(x)}{Q(x)}$ neprava, tada dijeljenjem polinoma $P(x)$ sa polinomom $Q(x)$ dobijemo količnik polinom $K(x)$, i ostatak dijeljenja, polinom $R(x)$, tako da je

$$P(x)=Q(x)\cdot K(x)+R(x) \quad \text{tj.} \quad \frac{P(x)}{Q(x)}=K(x)+\frac{R(x)}{Q(x)}$$

Kako je stepen polinoma ostatka $R(x)$ manji od stepena polinoma djelioca $Q(x)$, slijedi da je $\frac{R(x)}{Q(x)}$ prava racionalna f-ja.

Pravu racionalnu f-ju možemo integrirati metodom neodređenih koeficijenata. To činimo na sljedeći način:

Najprije polinom u nazivniku racionalne f-je rastavimo na proste faktore oblika

$$(x-a)^n \quad \text{i} \quad (x^2+px+q)^n \quad \text{gdje su}$$

$n\in N$, $a, p, q\in R$, $p^2-4q<0$.

Svakom faktoru oblika $(x-a)^n$ pridružimo f-ju oblika

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\dots+\frac{A_n}{(x-a)^n}$$

($A_1, A_2, \dots, A_n$ su konstante koje trebamo odrediti),

a svakom faktoru oblika $(x^2+px+q)^n$ pridružimo f-ju oblika

$$\frac{M_1x+N_1}{x^2+px+q}+\frac{M_2x+N_2}{(x^2+px+q)^2}+\dots+\frac{M_nx+N_n}{(x^2+px+q)^n}$$

gdje su $M_1, M_2, \dots, M_n$ i $N_1, N_2, \dots, N_n$ konstante koje treba odrediti.

Navedene racionalne f-je nazivamo parcijalni razlomci. Prema tome, svaku pravu racionalnu f-ju najprije rastavimo na zbir parcijalnih razlomaka, a zatim svaki od njih posebno integriramo. Kod neprave racionalne f-je najprije vršimo dijeljenje polinoma u brojniku sa polinomom u nazivniku,

Odrediti integrale

a) $\displaystyle\int \frac{3x^2+8}{x^3+4x^2+4x}\,dx$; b) $\displaystyle\int \frac{2x^5+6x^3+1}{x^4+3x^2}\,dx$;

c) $\displaystyle\int \frac{x^3+4x^2+4x}{x^4+x}\,dx$; d) $\displaystyle\int \frac{(x^3-3)\,dx}{x^4+10x^2+25}$.

Rj.

a) $\displaystyle\int \frac{3x^2+8}{x^3+4x^2+4x}\,dx$

Nazivnik rastavljamo na faktore.

$$x^3+4x^2+4x = x(x^2+4x+4) = x(x+2)^2$$

Poslije ovoga podintegralnu f-ju rastavljamo na sumu

$$\frac{3x^2+8}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

Da bi odredili koeficijente A, B, C čitavu jednakost množimo sa $x(x+2)^2$ nakon čega dobijamo

$$3x^2+8 = A\underbrace{(x+2)^2}_{x^2+2x+2} + Bx(x+2) + Cx$$

$$= (A+B)x^2 + (4A+2B+C)x + 4A$$

Sad izjednačavamo koeficijente koji stoje uz x^2, x i x^0.

x^2: $A+B=3$ $\Rightarrow B=1$

x: $4A+2B+C=0$ $\Rightarrow C=-10$

x^0: $4A=8$ $\Rightarrow A=2$

Prema tome dobili smo $\displaystyle\frac{3x^2+8}{x(x+2)^2} = \frac{2}{x} + \frac{1}{x+2} - \frac{10}{(x+2)^2}$

Sad možemo riješiti dati integral

$$\int \frac{3x^2+8}{x^3+4x^2+4x}\,dx = \int\left(\frac{2}{x} + \frac{1}{x+2} - \frac{10}{(x+2)^2}\right)dx =$$

$$= 2\int\frac{dx}{x} + \int\frac{dx}{x+2} - 10\int (x+2)^{-2}\,d(x+2) =$$

$$= 2\ln|x| + \ln|x+2| + \frac{10}{x+2} + C$$

b) $\displaystyle\int \frac{2x^5+6x^3+1}{x^4+3x^2}\,dx$

Podjelimo $2x^5+6x^3+1$ sa x^4+3x^2.

$$(2x^5+6x^3+1) : (x^4+3x^2) = 2x$$
$$\underline{-\;2x^5+6x^3}$$
$$1$$

Prema tome

$$\frac{2x^5+6x^3+1}{x^4+3x^2} = 2x + \frac{1}{x^4+3x^2}$$

Rastavimo nazivnik na faktore

$$x^4+3x^2 = x^2(x^2+3)$$

Napišimo ostatak $\displaystyle\frac{1}{x^4+3x^2}$ u obliku sume

$$\frac{1}{x^2(x^2+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3}$$

Da bi smo odredili koeficijente A, B, C i D pomnožimo dobijenu jednakost sa $x^2(x^2+3)$:

$$1 = Ax(x^2+3) + B(x^2+3) + (Cx+D)x^2 =$$

$$= (A+C)x^3 + (B+D)x^2 + 3Ax + 3B$$

Izjednačimo brojeve koji stoje uz x^3, x^2, x i x^0:

x^3: $A+C=0$ $\Rightarrow C=0$

x^2: $B+D=0$ $\Rightarrow D=-\frac{1}{3}$

x: $3A=0$ $\Rightarrow A=0$

x^0: $3B=1$ $\Rightarrow B=\frac{1}{3}$

$$\frac{1}{x^2(x^2+3)}=\frac{1}{3x^2}-\frac{1}{3(x^2+3)}$$

Sad nije teško izračunati dati integral

$$\int\frac{2x^5+6x^3+1}{x^4+3x^2}dx=\int\left(2x+\frac{1}{x^4+3x^2}\right)dx=$$

$$=\int\left(2x+\frac{\frac{1}{3}}{x^2}-\frac{\frac{1}{3}}{x^2+3}\right)dx=2\int x\,dx+\frac{1}{3}\int x^{-2}dx$$

$$-\frac{1}{3}\int\frac{dx}{x^2+3}=x^2-\frac{1}{3x}-\frac{1}{3\sqrt{3}}\operatorname{arc\,tg}\frac{x}{\sqrt{3}}+C$$

c) $\displaystyle\int\frac{x^3+4x^2+4x}{x^4+x}dx$

Rastavimo nazivnik x^4+x na faktore

$$x^4+x=x(x^3+1)=x(x+1)(x^2-x+1)$$

$$\frac{x^3+4x^2-2x+1}{x(x+1)(x^2-x+1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}\quad /\cdot x(x+1)(x^2-x+1)$$

$$x^3+4x^2-2x+1=A(x^3+1)+Bx(x^2-x+1)+(Cx+D)(x^2+x)$$

$$=(A+B+C)x^3+(C+D-B)x^2+(B+D)x+A$$

$A+B+C=1$

$-B+C+D=4$

$B+D=-2$

$A=1$

$A=1;\ B=-2;\ C=2;\ D=0$

$$\frac{x^3+4x^2-2x+1}{x(x+1)(x^2-x+1)}=\frac{1}{x}-\frac{2}{x+1}+\frac{2x}{x^2-x+1}$$

Sad nije teško odrediti dati integral

$$I=\int\frac{x^3+4x^2-2x+1}{x^4+x}dx=\int\frac{dx}{x}-2\int\frac{dx}{x+1}+2\int\frac{x\,dx}{x^2-x+1}$$

$$=\ln|x|-2\ln|x+1|+2I_1$$

Da bi odredili I_1 imamo

$$x^2-x+1=x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}$$

$$I_1=\int\frac{x\,dx}{(x-\frac{1}{2})^2+\frac{3}{4}}=\left|\begin{matrix}x-\frac{1}{2}=t\\ dx=dt\end{matrix}\right|=\int\frac{(t+\frac{1}{2})dt}{t^2+\frac{3}{4}}=\frac{1}{2}\int\frac{d(t^2+\frac{3}{4})}{t^2+\frac{3}{4}}$$

$$+\frac{1}{2}\int\frac{dt}{t^2+\frac{3}{4}}=\frac{1}{2}\ln\left(t^2+\frac{3}{4}\right)+\frac{1}{\sqrt{3}}\operatorname{arc\,tg}\frac{2t}{\sqrt{3}}=$$

$$=\frac{1}{2}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\operatorname{arc\,tg}\frac{2x-1}{\sqrt{3}}.$$

Prema tome imamo

$$I=\ln\frac{|x|(x^2-x+1)}{(x+1)^2}+\frac{2}{\sqrt{3}}\quad\frac{2x-1}{\sqrt{3}}+C.$$

d) $$\int \frac{(x^3-3)\,dx}{x^4+10x^2+25}$$

$$x^4+10x^2+25=(x^2+5)^2$$

$$\frac{x^3-3}{(x^2-5)^2}=\frac{Ax+B}{x^2+5}+\frac{Cx+D}{(x^2+5)^2} \quad /\cdot(x^2-5)^2$$

$$x^3-3=(Ax+B)(x^2+5)+(Cx+D)$$

$$=Ax^3+Bx^2+(5A+C)x+(5B+D)$$

$$\begin{aligned} A&=1\\ B&=0\\ 5A+C&=0\\ 5B+D&=-3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A&=1\\ B&=0\\ C&=-5\\ D&=-3 \end{aligned}$$

$$\frac{x^3-3}{(x^2+5)^2}=\frac{x}{x^2+5}-\frac{5x+3}{(x^2+5)^2}$$

Sad imamo

$$\int\frac{x^3-3}{x^4+10x^2+25}\,dx=\underbrace{\int\frac{x\,dx}{x^2+5}}_{I_1}-5\underbrace{\int\frac{x\,dx}{(x^2+5)^2}}_{I_2}-3\underbrace{\int\frac{dx}{(x^2+5)^2}}_{I_3}$$

$$I_1=\int\frac{x\,dx}{x^2+5}=\frac{1}{2}\int\frac{d(x^2+5)}{x^2+5}=\frac{1}{2}\ln|x^2+5|$$

$$I_2=\int\frac{x\,dx}{(x^2+5)^2}=\frac{1}{2}\int(x^2+5)^{-2}d(x^2+5)=\frac{1}{2}\cdot\frac{(x^2+5)^{-1}}{-1}=$$

$$=-\frac{1}{2(x^2+5)}$$

$$I_3=\int\frac{dx}{(x^2+5)^2}=\left|\begin{array}{l} x=\sqrt{5}\,\mathrm{tg}\,z=\sqrt{5}\,\frac{\sin z}{\cos z}\\ dx=\sqrt{5}\,\frac{\cos^2 z+\sin^2 z}{\cos^2 z}dz=\frac{\sqrt{5}}{\cos^2 z}dz\\ x^2=5\,\frac{1-\cos^2 z}{\cos^2 z}\end{array}\right|=$$

$$=\int\frac{\frac{\sqrt{5}\,dz}{\cos^2 z}}{\left(\frac{5-5\cos^2 z}{\cos^2 z}+5\right)^2}=\sqrt{5}\int\frac{\frac{dz}{\cos^2 z}}{\frac{25}{\cos^4 z}}=\int\cos^2 z\,dz=$$

$$=\frac{\sqrt{5}}{25}\cdot\frac{1}{2}\int(1+\cos 2z)\,dz=\frac{\sqrt{5}}{50}\left(z+\frac{1}{2}\sin 2z\right)$$

$$=\frac{1}{10\sqrt{5}}\left(\mathrm{arc\,tg}\,\frac{x}{\sqrt{5}}+\frac{x\sqrt{5}}{x^2+5}\right)$$

Prema tome

$$I=\frac{1}{2}\ln(x^2+5)+\frac{5}{2(x^2+5)}-\frac{3}{10\sqrt{5}}\left(\mathrm{arc\,tg}\,\frac{x}{\sqrt{5}}+\frac{x\sqrt{5}}{x^2+5}\right)+C$$

$$=\frac{1}{2}\ln(x^2+5)+\frac{25-3x}{10(x^2+5)}-\frac{3}{10\sqrt{5}}\,\mathrm{arc\,tg}\,\frac{x}{\sqrt{5}}+C$$

traženo rješenje

Zadaci za vježbu

Odrediti integrale

1. $\displaystyle\int \frac{dx}{x^3-x^2}$

2. $\displaystyle\int \frac{dx}{x^3+x}$

3. $\displaystyle\int \frac{x\,dx}{x^3-1}$

4. $\displaystyle\int \frac{(x^2+1)\,dx}{x^3-3x^2+3x-1}$

5. $\displaystyle\int \frac{(7x-15)\,dx}{x^3-2x^2+5x}$

6. $\displaystyle\int \frac{2t^5-2t+1}{1-t^4}\,dt$

7. $\displaystyle\int \frac{z^2\,dz}{z^4+5z^2+4}$

8. $\displaystyle\int \frac{x^4\,dx}{x^4-2x^2+1}$

9. * $\displaystyle\int \frac{(x+1)\,dx}{x^4+4x^2+4}$

10. * $\displaystyle\int \frac{1-x^4}{1+x^4}\,dx$

Rješenja:

1. $\frac{1}{x}+\ln\left|1-\frac{1}{x}\right|$ 2. $\ln\frac{|x|}{\sqrt{x^2+1}}$ 3. $\frac{1}{3}\ln\frac{|x-1|}{\sqrt{x^2+x+1}}+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$

4. $\ln|x-1|-\frac{2x-1}{(x-1)^2}$ 5. $3\ln\frac{\sqrt{x^2-2x+5}}{|x|}+2\operatorname{arctg}\frac{x-1}{2}$

6. $\frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|-t^2+\frac{1}{2}\operatorname{arctg}t$ 7. $\frac{2}{3}\operatorname{arctg}\frac{z}{2}-\frac{1}{3}\operatorname{arctg}z$

8. $\frac{2x^3-3x}{2(x^2-1)}+\frac{3}{4}\ln\left|\frac{x-1}{x+1}\right|$ 9. $\frac{x-2}{4(x^2+2)}+\frac{1}{4\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}$

10. $-x+\frac{1}{2\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\operatorname{arctg}\frac{x\sqrt{2}}{1-x^2}$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija racionalnih funkcija)

$q_m(x)$ - polinom stepena m

npr. $p(x)=5x^7-3x^2+x+8$, polinom 7-og stepena

Racionalna f-ja je količnik dva polinoma.

$$s(x)=\frac{p_n(x)}{q_m(x)}.$$ Za $n<m$ $s(x)$ je prava racionalna f-ja

Racionalnu f-ju razlažemo na proste razlomke.
Prosti razlomci su oblika:

$$\frac{A}{(ax+b)^n},\quad \frac{Bx+C}{(ax^2+bx+c)^n},\quad n\in\mathbb{N}$$

Izračunajte integrale:

1. $$I=\int\frac{x}{(x-1)(x+1)^2}\,dx$$

Rj. $$\frac{x}{(x-1)(x+1)^2}=\frac{a}{x-1}+\frac{b}{x+1}+\frac{c}{(x+1)^2}\quad /\cdot(x-1)(x+1)^2$$

$$x=a(x+1)^2+b(x-1)(x+1)+c(x-1)$$
$$x=a(x^2+2x+1)+b(x^2-1)+c(x-1)$$
$$x=(a+b)x^2+(2a+c)x+(a-b-c)$$

$$\Rightarrow\quad \begin{aligned} a+b\phantom{{}+c}&=0 &(1)\\ 2a\phantom{{}-b}+c&=1 &(2)\\ a-b-c&=0 &(3)\end{aligned}$$

$$\begin{aligned}(1)\quad a+b&=0\\ (2)+(3):\ 3a-b&=1\\ \hline 4a&=1\end{aligned}\qquad a=\frac{1}{4},\quad b=-\frac{1}{4},\quad c=\frac{1}{2}$$

$$\frac{x}{(x-1)(x+1)^2}=\frac{\frac{1}{4}}{x-1}+\frac{-\frac{1}{4}}{x+1}+\frac{\frac{1}{2}}{(x+1)^2}$$

$$I=\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{(x+1)^2}=$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|+\frac{1}{2}\,\frac{(x+1)^{-1}}{-1}+C=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2(x+1)}+C$$

2. $I=\int\frac{8x-31}{x^2-9x+14}\,dx$,

$x^2-9x+14=0$ $\quad x_{1,2}=\frac{9\pm5}{2}$

$D=81-56$ $\quad x_1=2\ ,\ x_2=7$

$D=25$ $\quad x^2-9x+14=(x-2)(x-7)$

Rj. $\frac{8x-31}{x^2-9x+14}=\frac{8x-31}{(x-2)(x-7)}=\frac{a}{x-2}+\frac{b}{x-7}\quad /(x-2)(x-7)$

$8x-31=a(x-7)+b(x-2)$

$8x-31=(a+b)x+(-7a-2b)$

$$\begin{array}{l} a+b=8 \quad /\cdot 2\\ -7a-2b=-31\\ \hline 2a+2b=16\\ -7a-2b=-31\\ \hline \end{array}\qquad \begin{array}{l} -5a=-15\\ a=3\\ b=5\end{array}$$

$$\frac{8x-31}{x^2-9x+14}=\frac{3}{x-2}+\frac{5}{x-7}$$

$$I=3\int\frac{dx}{x-2}+5\int\frac{dx}{x-7}=3\ln|x-2|+5\ln|x-7|+C$$

3. $I=\int\frac{dx}{x^3-2x^2+x}$, $\quad x^3-2x^2+x=x(x^2-2x+1)=x(x-1)^2$

Rj. $\frac{1}{x^3-2x^2+x}=\frac{1}{x(x-1)^2}=\frac{a}{x}+\frac{b}{x-1}+\frac{c}{(x-1)^2}\quad /x(x-1)^2$

$1=a(x-1)^2+bx(x-1)+cx$

$1=a(x^2-2x+1)+b(x^2-x)+cx$

$1=(a+b)x^2+(-2a-b+c)x+a$

$$\begin{array}{l} a+b=0\\ -2a-b+c=0\\ a=1\\ \hline b=-1 \Rightarrow c=1\end{array}$$

$$\frac{1}{x^3-2x^2+x}=\frac{1}{x}+\frac{-1}{x-1}+\frac{1}{(x-1)^2}$$

$$I=\int\frac{dx}{x}-\int\frac{dx}{x-1}+\int\frac{dx}{(x-1)^2}=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C$$

4. $I=\int\frac{x^3+x+1}{x^4-1}\,dx$, $\quad x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$

Rj. $\frac{x^3+x+1}{x^4-1}=\frac{a}{x-1}+\frac{b}{x+1}+\frac{cx+d}{x^2+1}\quad /(x-1)(x+1)(x^2+1)$

$x^3+x+1=a(x+1)(x^2+1)+b(x-1)(x^2+1)+(cx+d)(x-1)(x+1)$

$x^3+x+1=a(x^3+x+x^2+1)+b(x^3+x-x^2-1)+(cx+d)(x^2-1)$

$x^3+x+1=a(x^3+x^2+x+1)+b(x^3-x^2+x-1)+c(x^3-x)+d(x^2-1)$

$x^3+x+1=(a+b+c)x^3+(a-b+d)x^2+(a+b-c)x+(a-b-d)$

$$\begin{array}{ll} a+b+c=1 & (1)\\ a-b+d=0 & (2)\\ a+b-c=1 & (3)\\ a-b-d=1 & (4)\\ \hline \end{array}$$

$(1)-(4):\ 2b+c+d=0$

$(2)-(4):\ 2d=-1 \Rightarrow d=-\frac{1}{2}$

$(3)-(4):\ 2b-c+d=0$

$$\begin{array}{l} 2b+c=\frac{1}{2}\\ -\ \ 2b-c=\frac{1}{2}\\ \hline 2c=0 \Rightarrow c=0\end{array}$$

$2b=\frac{1}{2}$

$b=\frac{1}{4}$

$a=1-\frac{1}{4}-0$

$a=\frac{3}{4}$

$$\frac{x^3+x+1}{x^4-1}=\frac{\frac{3}{4}}{x-1}+\frac{\frac{1}{4}}{x+1}+\frac{-\frac{1}{2}}{x^2+1}$$

$$I=\frac{3}{4}\int\frac{dx}{x-1}+\frac{1}{4}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{dx}{x^2+1}=\frac{3}{4}\ln|x-1|+\frac{1}{4}\ln|x+1|-\frac{1}{2}\operatorname{arc\,tg}x+C$$

5. $\int\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}\,dx$,

$x^2+1=0$ $\quad x^2-x+1=0$

$D=-4<0$ $\quad D=1-4<0$

x^2+1 ; x^2-x+1 se ne mogu rastaviti

Rj. $\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}=\frac{ax+b}{x^2+1}+\frac{cx+d}{x^2-x+1}\quad /(x^2+1)(x^2-x+1)$

$x^3+3x-5=(ax+b)(x^2-x+1)+(cx+d)(x^2+1)$

$x^3+3x-5=a(x^3-x^2+x)+b(x^2-x+1)+c(x^3+x)+d(x^2+1)$

$x^3+3x-5=(a+c)x^3+(-a+b+d)x^2+(a-b+c)x+(b+d)$

$$\begin{array}{ll} a+c=1 & (1)\\ -a+b+d=0 & (2)\\ a-b+c=3 & (3)\\ b+d=-5 & (4)\\ \hline \end{array}$$

$(1)\ a+c=1$

$(2)-(4)\ -a=5 \Rightarrow a=-5$

$(3)\ a-b+c=3$

$c=6$

$-5-b+6=3 \Rightarrow b=-2$

$d=-3$

$$\frac{x^3-3x-5}{(x^2+1)(x^2-x+1)}=\frac{-5x-2}{x^2+1}+\frac{6x-3}{x^2-x+1}$$

$(x^2+1)'=2x$

$-5x-2=\alpha\cdot 2x+\beta$

$-5x-2=2\cdot\frac{-5}{2}x-2$

$$I=\int\frac{-5x-2}{x^2+1}\,dx+3\int\frac{2x-1}{x^2-x+1}\,dx=I_1+3I_2$$

$$I_1 = \int \frac{2\cdot\frac{-5}{2}x - 2}{x^2+1}\,dx = -\frac{5}{2}\int\frac{2x}{x^2+1}\,dx - 2\int\frac{dx}{x^2+1} = -\frac{5}{2}\ln|x^2+1| - 2\,\mathrm{arctg}(x^2+1) + C_1$$

$$I_2 = \int \frac{2x-1}{x^2-x+1}\,dx = \left|\begin{matrix} x^2-x+1=t \\ (2x-1)dx=dt\end{matrix}\right| = \int\frac{dt}{t} = \ln|x^2-x+1| + C_2$$

$$I = -\frac{5}{2}\ln|x^2+1| - 2\,\mathrm{arctg}(x^2+1) + 3\ln|x^2-x+1| + C$$

6. $I = \int \frac{3}{x(x+1)^3}\,dx$

Rj. $$\frac{3}{x(x+1)^3} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2} + \frac{d}{(x+1)^3} \quad /\cdot x(x+1)^3$$

$$3 = a(x+1)^3 + bx(x+1)^2 + cx(x+1) + dx$$

Ako uvrstimo $x=0$ u gornju jednakost dobićemo $3=a$

tj. $a=3$: $\quad 3 = 3(x+1)^3 + bx(x+1)^2 + cx(x+1) + dx$

za $x=-1$ imamo $\quad 3 = d\cdot(-1) \Rightarrow d=-3$

$$3 = 3(x+1)^3 + bx(x+1)^2 + cx(x+1) - 3x$$

za $x=1$: $\quad 3 = 3\cdot 2^3 + b\cdot 1\cdot 2^2 + c\cdot 1\cdot 2 - 3\cdot 1 \Rightarrow 4b+2c=-18$

za $x=-2$: $\quad 3 = 3\cdot(-1)^3 + b(-2)(-1)^2 + c(-2)(-1) - 3\cdot(-2) \Rightarrow -2b+2c=0$ $\Big\}+$

$\Rightarrow b=c=-3$

$$\int\frac{dx}{(x+1)^3} = \int(x+1)^{-3}\,dx = \frac{(x+1)^{-2}}{-2} + C_1 = -\frac{1}{2(x+1)^2} + C_1$$

$$\frac{3}{x(x+1)^3} = \frac{3}{x} + \frac{-3}{x+1} + \frac{-3}{(x+1)^2} + \frac{-3}{(x+1)^3}$$

$$I = 3\int\frac{dx}{x} - 3\int\frac{dx}{x+1} - 3\int\frac{dx}{(x+1)^2} - 3\int\frac{dx}{(x+1)^3} = 3\ln|x| - 3\ln|x+1| + \frac{3}{x+1} + \frac{3}{2(x+1)^2} + C$$

7.✓ $\int\frac{2x-3}{(x^2-3x+2)^3}\,dx$ $\quad$ Rj. $-\frac{1}{2(x^2-3x+2)^2} + C$

8.✓ $\int\frac{x^3+x+1}{x(x^2+1)}\,dx$ $\quad$ Rj. $x + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

9.✓ $\int\frac{x^4}{x^4-1}\,dx$ $\quad$ Rj. $x + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\,\mathrm{arctg}\,x + C$

10.✓ $\int\frac{dx}{(x^2-4x+3)(x^2+4x+5)}$ $\quad$ Rj: $\frac{1}{52}\ln|x-3| - \frac{1}{20}\ln|x-1| + \frac{1}{65}\ln(x^2+4x+5) + \frac{7}{130}\,\mathrm{arctg}(x+2) + C$

11. $I = \int\frac{-x^5-2x^2+2}{(x-1)(x^2+2x+1)}\,dx$

Rj. $(x-1)(x^2+2x+1) = x^3+2x^2+x-x^2-2x-1 = x^3+x^2-x-1$

$$(-x^5-2x^2+2):(x^3+x^2-x-1) = -x^2+x-2 - \frac{x}{x^3+x^2-x-1}$$

$-(-x^5-x^4+x^3+x^2)$

$= x^4-x^3-3x^2+2$

$-(x^4+x^3-x^2-x)$

$= -2x^3-2x^2+x+2$

$-(-2x^3-2x^2+2x+2)$

$-x$

$$I = \int\left(-x^2+x-2 - \frac{x}{(x-1)(x^2+2x+1)}\right)dx$$

$$= -\int x^2\,dx + \int x\,dx - 2\int dx - \int\frac{x}{(x-1)(x+1)^2}\,dx$$

$= -\frac{x^3}{3} + \frac{x^2}{2} - 2x - I_1$, integral I_1 smo odredili u zadatku 1

$$I = -\frac{x^3}{3} + \frac{x^2}{2} - 2x - \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2(x+1)} + C$$

12. $I = \int\frac{x^5-2x^3+x-1}{x^3-2x^2+x}\,dx$

Rj. $$(x^5-2x^3+x-1):(x^3-2x^2+x) = x^2+2x+1 - \frac{1}{x^3-2x^2+x}$$

$-(x^5-2x^4+x^3)$

$= 2x^4-3x^3+x-1$

$-(2x^4-4x^3+2x^2)$

$= x^3-2x^2+x-1$

$-(x^3-2x^2+x)$

-1

$$I = \int\left(x^2+2x+1 - \frac{1}{x^3-2x^2+x}\right)dx$$

$$= \int x^2\,dx + 2\int x\,dx + \int dx - \int\frac{dx}{x(x-1)^2}$$

$= \frac{x^3}{3} + 2\cdot\frac{x^2}{2} + x - I_2$, integral I_2 smo odredili u zadatku 3.

$$I = \frac{x^3}{3} + x^2 + x - \ln|x| + \ln|x-1| + \frac{1}{x-1} + C$$

13.✓ $I = \int\frac{x^5+2x^3-4}{(x-2)(x^2+x+1)}\,dx$ $\quad$ Rj. $I = \frac{x^3}{3} + \frac{x^2}{2} + 4x + \frac{44}{7}\ln|x-2| + \frac{5}{14}\ln|x^2+x+1| + \frac{11}{7\sqrt{3}}\,\mathrm{arctg}\frac{2x+1}{\sqrt{3}} + C$

14. $\displaystyle\int \frac{x^5-60x^3+73x^2+171}{x^2-9x+14}\,dx$

Rj. $(x^5-60x^3+73x^2+171):(x^2-9x+14) = x^3+9x^2+7x+10+\dfrac{-8x+31}{x^2-9x+14}$

$$\begin{array}{l}
-\ \underline{x^5-9x^4+14x^3} \\
=\ \ 9x^4-74x^3+73x^2+171 \\
-\ \ \underline{9x^4-81x^3+126x^2} \\
=\ \ \ \ 7x^3-53x^2+171 \\
-\ \ \ \ \underline{7x^3-63x^2+98x} \\
=\ \ \ \ \ \ 10x^2-98x+171 \\
-\ \ \ \ \ \ \underline{10x^2-90x+140} \\
\ \ \ \ \ \ \ \ \ -8x+31
\end{array}$$

$$I=\int\left(x^3+9x^2+7x+10+\frac{-8x+31}{x^2-9x+14}\right)dx$$
$$=\int x^3dx+9\int x^2dx+7\int x\,dx+10\int dx-\int\frac{8x-31}{x^2-9x+14}\,dx$$

$I=\dfrac{x^4}{4}+9\cdot\dfrac{x^3}{3}+7\dfrac{x^2}{2}+10x-I_3$, integral I_3 smo odredili u zadatku broj 2.

$I=\frac14 x^4+3x^3+\frac72 x^2+10x-3\ln|x-2|+5\ln|x-7|+C$

15. $\displaystyle I=\int\frac{x^7-2x^6+x^5+x^4+2x^2}{x^4-1}\,dx$

Rj. $(x^7-2x^6+x^5+x^4+2x^2):(x^4-1)=x^3-2x^2+x+1+\dfrac{x^3+x+1}{x^4-1}$

$$\begin{array}{l}
-\ \underline{x^7-x^3} \\
=\ \ -2x^6+x^5+x^4+x^3+2x^2 \\
-\ \ \underline{-2x^6+2x^2} \\
=\ \ \ \ x^5+x^4+x^3 \\
\ \ \ \ \ \ \underline{x^5-x} \\
\ \ \ \ \ \ \ \ x^4+x^3+x \\
-\ \ \ \ \ \ \underline{x^4-1} \\
\ \ \ \ \ \ \ \ \ \ x^3+x+1
\end{array}$$

$$I=\int\left(x^3-2x^2+x+1+\frac{x^3+x+1}{x^4-1}\right)dx=$$
$$=\int x^3dx-2\int x^2dx+\int x\,dx+\int dx+\int\frac{x^3+x+1}{x^4-1}\,dx$$

$I=\dfrac{x^4}{4}-2\cdot\dfrac{x^3}{3}+\dfrac{x^2}{2}+x+I_4$, integral I_4 smo odredili u zadatku 4

$I=\frac14x^4-\frac23x^3+\frac12x^2+x+\frac34\ln|x-1|+\frac14\ln|x+1|-\frac12\operatorname{arctg}x+C$

16.ˇ $\displaystyle\int\frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}\,dx$ Rj. $I=\dfrac{x^2}{2}-2x+\dfrac{2}{x}+2\ln|x^2+2x+2|-$
$-2\operatorname{arctg}(x+1)+C$

17. $\displaystyle\int\frac{-2x^7-x^6-x^3+6x^2-x}{(x^2+1)(x^2-x+1)}\,dx$

Rj. $(x^2+1)\cdot(x^2-x+1)=x^4-x^3+x^2+x^2-x+1=x^4-x^3+2x^2-x+1$

$(-2x^7-x^6-x^3+6x^2-x):(x^4-x^3+2x^2-x+1)=-2x^3-3x^2+x+5+\dfrac{x^3+3x-5}{x^4-x^3+2x^2-x+1}$

$$\begin{array}{l}
-\ \underline{-2x^7+2x^6-4x^5+2x^4-2x^3} \\
=\ \ -3x^6+4x^5-2x^4+x^3+6x^2-x \\
-\ \ \underline{-3x^6+3x^5-6x^4+3x^3-3x^2} \\
=\ \ \ \ x^5+4x^4-2x^3+9x^2-x \\
-\ \ \ \ \underline{x^5-x^4+2x^3-x^2+x} \\
=\ \ \ \ \ \ 5x^4-4x^3+10x^2-2x \\
-\ \ \ \ \ \ \underline{5x^4-5x^3+10x^2-5x+5} \\
\ \ \ \ \ \ \ \ \ \ x^3+3x-5
\end{array}$$

$$I=\int\left(-2x^3-3x^2+x+5+\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}\right)dx$$
$$=-2\int x^3dx-3\int x^2dx+\int x\,dx+5\int dx+\int\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}\,dx$$

$I=-2\cdot\dfrac{x^4}{4}-3\cdot\dfrac{x^3}{3}+\dfrac{x^2}{2}+5x+I_5$, integral I_5 smo odredili u zadatku broj 5

$I=-\frac12x^4-x^3+\frac12x^2+5x-\frac52\ln|x^2+1|-2\operatorname{arctg}(x^2+1)+3\ln|x^2-x+1|+C$

18.ˇ $\displaystyle\int\frac{x^5+x^4-8}{x^3-4x}$ 19.ˇ $\displaystyle\int\frac{x^5+2}{x^3-1}\,dx$

8. Integracija nekih iracionalnih f-ja

Ovu lekciju možemo podjeliti na pet vrsta integrala

I. $\int R(x, x^{\alpha}, x^{\beta}, \dots)\,dx$ gdje je R racionalna f-ja,

$\alpha = \frac{m_1}{n_1}$, $\beta = \frac{m_2}{n_2}$ - uvodimo smjenu $x = t^k$ gdje je k broj takav da u novodobijenom integralu na promjenjivoj t ostanu samo cijeli stepeni;

$\int R(x, (ax+b)^{\alpha}, (ax+b)^{\beta})\,dx$ ili

$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\alpha}, \left(\frac{ax+b}{cx+d}\right)^{\beta}, \dots\right)$

rješavamo uvođenjem smjene $ax+b = t^k$ ili

$\frac{ax+b}{cx+d} = t^k$.

II. $\int R(x, \sqrt{a^2-x^2})\,dx$ - uvodimo smjenu $x = a\sin t$;

$\int R(x, \sqrt{a^2+x^2})\,dx$ - uvodimo smjenu $x = a\,tg\,t$;

$\int R(x, \sqrt{x^2-a^2})\,dx$ - uvodimo smjenu $x = \frac{a}{\cos t}$

III. $\int x^m (a+bx^n)^p\,dx$ (integral binomnog diferencijala)

a) kada je $p \in \mathbb{Z}$ (p cijeli broj) - uvodimo smjenu $x = t^s$ gdje je $s = NZS(m, n)$ (najmanji zajednički sadržalac)

ili (ako je $p \in \mathbb{Z}$) razlažemo na dijelove pomoću binomne formule

b) kada je $\frac{m+1}{n} \in \mathbb{Z}$ - uvodimo smjenu $a+bx^n = t^r$ gdje je r nazivnik od p

c) kada je $\frac{m+1}{n} + p \in \mathbb{Z}$ - uvodimo smjenu $a+bx^n = x^n t^r$ gdje je r nazivnik broja p

IV. $\int \frac{P_n(x)}{\sqrt{V}}\,dx$ gdje je $P_n(x)$ polinom n-tog stepena, a $V = ax^2+bx+c$. Ovaj integral možemo odrediti po formuli

$$\int \frac{P_n(x)}{\sqrt{V}}\,dx = (A_1 x^{n-1} + A_2 x^{n-2} + \dots + A_n)\sqrt{V} + B\int \frac{dx}{\sqrt{V}}$$

gdje su $A_1, A_2, \dots, A_n, B$ brojevi koje dobijemo iz sistema jednačina, a sistem jednačina dobijemo tako što datu formulu prvo diferenciramo a onda dobijeni diferencijal pomnožimo sa $\sqrt{V}$. (Metoda Ostrogradski)

V $\int \frac{(Ax+B)\,dx}{(x-\alpha)\sqrt{ax^2+bx+c}}$ - uvodimo smjenu $x - \alpha = \frac{1}{t}$

Odrediti integrale

a) $\displaystyle\int \frac{1+\sqrt[4]{x}}{x+\sqrt{x}}\,dx$ b) $\displaystyle\int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx$ c) $\displaystyle\int \frac{\sqrt{(4-x^2)^3}}{x^6}\,dx$

d) $\displaystyle\int \frac{dx}{x^2\sqrt[3]{(1+x^3)^5}}$ e) $\displaystyle\int \frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx$ f) $\displaystyle\int \frac{dx}{(x-1)\sqrt{1-x^2}}$

Rj.

a) $$\int \frac{1+\sqrt[4]{x}}{x+\sqrt{x}}\,dx = \left| \begin{array}{l} x=t^4 \\ dx = 4t^3dt \end{array} \right| = \int \frac{1+t}{t^4+t^2}\,4t^3dt$$

$$=4\int \frac{t^2+t\overset{+1-1}{}}{t^2+1}\,dt = 4\int\left(1+\frac{t-1}{t^2+1}\right)dt =$$

$$=4\left(\int dt + \int \frac{\overbrace{\tfrac{1}{2}d(t^2+1)}^{=t\,dt}}{t^2+1} - \int\frac{dt}{t^2+1}\right) =$$

$$=4t+2\ln(t^2+1)-4\,\text{arc tg}\, t + C$$

$$=4\sqrt[4]{x}+2\ln(1+\sqrt{x})-4\,\text{arc tg}\sqrt[4]{x}+C$$

b) $\displaystyle\int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx$ Prema pravilu I: $\dfrac{1+x}{x}=t^2$

Odavde imamo $\dfrac{1\cdot x-(1+x)\cdot 1}{x^2}\,dx = 2t\,dt \Rightarrow \dfrac{-1}{x^2}\,dx = 2t\,dt$

$\Rightarrow \dfrac{dx}{x^2} = -2t\,dt$

$\dfrac{1+x}{x}=t^2 \Rightarrow 1+x=t^2\cdot x \Rightarrow (1-t^2)x=-1 \Rightarrow \boxed{x=\dfrac{1}{t^2-1}}$

$$\int\frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx = \left|\begin{array}{l}\frac{1+x}{x}=t^2 \\ \frac{dx}{x^2}=-2t\,dt\end{array}\right| = \int(-2)\,t\cdot t\,dt =$$

$$=-2\int t^2dt = -\frac{2}{3}t^3+C = -\frac{2}{3}\sqrt{\left(\frac{1+x}{x}\right)^3}+C$$

$\sqrt{4^3(1-\sin^2 t)^3}$

c) $$\int\frac{\sqrt{(4-x^2)^3}}{x^6}\,dx = \left|\begin{array}{l} x=2\sin t \\ dx = 2\cos t\,dt\end{array}\right| = \int \frac{\sqrt{(4-4\sin^2 t)^3}}{64\sin^6 t}\,2\cos t\,dt$$

$$=\frac{2^3\cdot 2}{64}\int\frac{\cos^3 t}{\sin^6 t}\cos t\,dt = \frac{1}{4}\int\frac{\cos^4 t}{\sin^6 t}\,dt = \frac{1}{4}\int \text{ctg}^4 t\cdot\frac{dt}{\sin^2 t}$$

$$=\left|\begin{array}{l} d(\text{ctg}\,t) = \frac{-dt}{\sin^2 t} \\ \frac{dt}{\sin^2 t} = -d(\text{ctg}\,t)\end{array}\right| = -\frac{1}{4}\int \text{ctg}^4 t\, d(\text{ctg}\,t) =$$

$$=-\frac{1}{4}\cdot\frac{1}{5}\text{ctg}^5 t + C = \left| \text{ctg}^5 t = \frac{\cos^5 t}{\sin^5 t} = \frac{(\cos^2 t)^{\frac{5}{2}}}{(\sin^2 t)^{\frac{5}{2}}} = \frac{(1-\sin^2 t)^{\frac{5}{2}}}{(\sin^2 t)^{\frac{5}{2}}} = \frac{\left(1-\frac{x^2}{4}\right)^{\frac{5}{2}}}{\left(\frac{x^2}{4}\right)^{\frac{5}{2}}} = \right.$$

$$\left. = \frac{(4-x^2)^{\frac{5}{2}}}{(x^2)^{\frac{5}{2}}} = \frac{\sqrt{(4-x^2)^5}}{x^5}\right| = C - \frac{\sqrt{(4-x^2)^5}}{20x^5}$$

d) $\displaystyle\int\frac{dx}{x^2\sqrt[3]{(1+x^3)^5}}$ ovo je integral binomnog diferencijala $m=-2,\ n=3,\ p=-\frac{5}{3}$

$\dfrac{m+1}{n} = \dfrac{-2+1}{3} = -\dfrac{1}{3}\notin\mathbb{Z}$ $\dfrac{m+1}{n}+p = -\dfrac{1}{3}-\dfrac{5}{3} = -\dfrac{6}{3} = -2\in\mathbb{Z}$

Prema pravilu III uvodimo smjenu $1+x^3 = x^3z^3$

$$1+x^3=x^3z^3$$
$$x^3z^3-x^3=1$$
$$(z^3-1)x^3=1$$
$$x^3=\frac{1}{z^3-1}$$
$$x=\frac{1}{(z^3-1)^{\frac{1}{3}}}$$

$$1+x^3=\frac{z^3}{z^3-1}$$
$$x^2\sqrt[3]{(1+x^3)^5}=\frac{1}{(z^3-1)^{\frac{2}{3}}}\sqrt[3]{\left(\frac{z^3}{z^3-1}\right)^5}=$$
$$=\frac{1}{(z^3-1)^{\frac{2}{3}}}\cdot\frac{z^5}{(z^3-1)^{\frac{5}{3}}}=\frac{z^5}{(z^3-1)^{\frac{7}{3}}}$$
$$dx=\left((z^3-1)^{-\frac{1}{3}}\right)'dz=-\frac{1}{3}\cdot(z^3-1)^{-\frac{4}{3}}\cdot 3z^2\,dz=$$
$$=\frac{-z^2}{(z^3-1)^{\frac{4}{3}}}\,dz$$

$$\int\frac{dx}{x^2\sqrt[3]{(1+x^3)^5}}=\left|\begin{matrix}1+x^3=x^3z^3\\ \vdots\end{matrix}\right|=\int\frac{-z^2}{(z^3-1)^{\frac{4}{3}}}\cdot\frac{(z^3-1)^{\frac{7}{3}}}{z^5}\,dz=$$
$$=-\int\frac{z^3-1}{z^2}\,dz=\int\frac{1-z^3}{z^3}\,dz=\int z^{-3}dz-\int dz=\frac{z^{-2}}{-2}-z+C$$
$$=-\frac{1}{2z^2}-z+C=\left|\begin{matrix}z^3=x^{-3}+1\\ z=\sqrt[3]{1+\frac{1}{x^3}}\end{matrix}\right|=-\frac{1}{2\sqrt[3]{\left(1+\frac{1}{x^3}\right)^2}}-\sqrt[3]{1+\frac{1}{x^3}}+C$$

e) $\displaystyle\int\frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx$ Prema pravilu $\underline{IV}$ (što je još poznato pod imenom metoda Ostrogradskog) imamo

$$I=\int\frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx=(Ax+B)\sqrt{x^2-2x}+D\int\frac{dx}{\sqrt{x^2-2x}}\quad\Big/\frac{d}{dx}$$
$$\frac{2x^2-x-5}{\sqrt{x^2-2x}}=A\sqrt{x^2-2x}+(Ax+B)\frac{2x-2}{2\sqrt{x^2-2x}}+\frac{D}{\sqrt{x^2-2x}}\quad\Big/\cdot\sqrt{x^2-2x}$$

$$2x^2-x-5=A(x^2-2x)+(Ax+B)(x-1)+D$$
$$2x^2-x-5=2Ax^2+(B-3A)x+(D-B)$$

Izjednačavamo koeficijente uz isti stepen

x^2: $2A=2 \Rightarrow A=1$

x: $B-3A=-1 \qquad B=2$

x^0: $D-B=-5 \qquad D=-3$

$$I=(x+2)\sqrt{x^2-2x}-3\int\frac{dx}{\sqrt{x^2-2x}}$$

$$x^2-2x=x^2-2\cdot x\cdot 1+1-1=(x-1)^2-1$$

$$\int\frac{dx}{\sqrt{x^2-2x}}=\int\frac{d(x-1)}{\sqrt{(x-1)^2-1}}=\ln\left|x-1-\sqrt{(x-1)^2-1}\right|$$

Prema tome
$$\int\frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx=(x+2)\sqrt{x^2-2x}-3\ln\left|x-1-\sqrt{(x-1)^2-1}\right|+C$$

f) $\displaystyle\int\frac{dx}{(x-1)\sqrt{1-x^2}}$ Prema pravilu $\underline{V}$ uvodimo smjenu $x-1=\frac{1}{t}$

$$dx=-\frac{1}{t^2}dt$$

$$I=\int\frac{dx}{(x-1)\sqrt{1-x^2}}=\left|\begin{matrix}x-1=\frac{1}{t}\\ dx=-\frac{1}{t^2}dt\end{matrix}\right|=\int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{1-\underbrace{\left(\frac{1}{t}+1\right)^2}_{\frac{1}{t^2}+\frac{2}{t}+1}}}=$$

$-1-2t>0$
$2t<-1$
$t<-\frac{1}{2}$
$|t|=-t$

$$=\int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{-\frac{1+2t}{t^2}}}=-\int\frac{\frac{dt}{t}}{\frac{\sqrt{-1-2t}}{|t|}}=-\int\frac{|t|\,dt}{t\sqrt{-1-2t}}$$

zato što je $\sqrt{t^2}=|t|$

$$=\int\frac{dt}{\sqrt{-1-2t}}=\int(-1-2t)^{-\frac{1}{2}}\left(-\frac{1}{2}\right)d(-1-2t)=-\frac{1}{2}\cdot\frac{(-1-2t)^{\frac{1}{2}}}{\frac{1}{2}}+C=C-\sqrt{-1-\frac{2}{x-1}}$$

Zadaci za vježbu

1. $\displaystyle\int \frac{dx}{(1+\sqrt[3]{x})\sqrt{x}}$ 2. $\displaystyle\int x\sqrt{3-x}\,dx$ 3. $\displaystyle\int \frac{1}{x}\sqrt{\frac{x-2}{x}}\,dx$

4. $\displaystyle\int \frac{dx}{\sqrt{(5-x^2)^3}}$ 5. $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\,dx$ 6. $\displaystyle\int x^2\sqrt{4-x^2}\,dx$

7.* $\displaystyle\int \frac{dx}{x\sqrt{x^2-9}}$ 8.* $\displaystyle\int \frac{\sqrt[3]{(1+2x^3)^2}}{x^6}\,dx$

9. $\displaystyle\int \frac{dt}{t\sqrt{1-t^3}}$ 10. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+2x+3}}$

11. $\displaystyle\int \frac{x^2+4x}{\sqrt{x^2+2x+2}}\,dx$ 12.* $\displaystyle\int \frac{x^2\,dx}{\sqrt{2ax-x^2}}$

Rješenja:

1. $6\sqrt[6]{x}-6\,\mathrm{arctg}\sqrt[6]{x}$ 2. $0{,}4(x^3-x-6)\sqrt{3-x}$

3. $-2\sqrt{\frac{x-2}{x}}-\ln\left[|x|\left(1-\sqrt{\frac{x-2}{x}}\right)^2\right]$ 4. $\frac{x}{5\sqrt{5-x^2}}$ 5. $\ln(x+\sqrt{x^2+1})$
$-\frac{\sqrt{x^2+1}}{x}$ 6. $2\arcsin\frac{x}{2}+\frac{x}{4}(x^2-2)\sqrt{4-x^2}$ 7. $\pm\frac{1}{3}\arccos\frac{3}{x}$
gdje je + kad $x>0$, − kad $x<0$. 8. $-\frac{1}{5}x^{-5}(2x^3+1)^{\frac{5}{3}}$

9. $\frac{1}{3}\ln\frac{|1-\sqrt{1-x^3}|}{1+\sqrt{1-x^3}}$ 10. $\frac{x-3}{2}\sqrt{x^2+2x+3}$ 11. $\frac{x+5}{2}\sqrt{x^2+2x+2}-$
$-\frac{7}{2}\ln(x+1+\sqrt{x^2+2x+2})$ 12. $\frac{3a^2}{2}\arcsin\frac{x-a}{a}-\frac{x+3a}{2}\sqrt{2ax-x^2}$

Izabrani **Zadaci za vježbu** sa rješenjima
(iz lekcije Integracija nekih iracionalnih funkcija)

Metoda Ostrogradskog

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c}+\lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

1. $I=\displaystyle\int\frac{3x^3}{\sqrt{x^2+4x+5}}\,dx$ Rj: $=(ax^2+bx+c)\sqrt{x^2+4x+5}+\lambda\displaystyle\int\frac{dx}{\sqrt{x^2+4x+5}}$ $\Big/\frac{d}{dx}$

$$\frac{3x^3}{\sqrt{x^2+4x+5}}=(2ax+b)\sqrt{x^2+4x+5}+(ax^2+bx+c)\cdot\frac{2x+4}{2\sqrt{x^2+4x+5}}+\lambda\cdot\frac{1}{\sqrt{x^2+4x+5}}\quad\Big/\cdot\sqrt{x^2+4x+5}$$

$$3x^3=(2ax+b)(x^2+4x+5)+(ax^2+bx+c)(x+2)+\lambda$$
$$3x^3=(2ax^3+8ax^2+10ax)+b(x^2+4x+5)+(ax^3+bx^2+cx)+2ax^2+2bx+2c+\lambda$$
$$3x^3=(2a+a)x^3+(8a+b+b+2a)x^2+(10a+4b+c+2b)x+5b+2c+\lambda$$
$$3x^3=3ax^3+(10a+2b)x^2+(10a+6b+c)x+5b+2c+\lambda$$

uz x^3: $3a=3 \Rightarrow a=1$
uz x^2: $10a+2b=0 \Rightarrow b=-5$
uz x: $10a+6b+c=0 \Rightarrow c=20$
uz x^0: $5b+2c+\lambda=0$ $\lambda=-15$

$$I=(x^2-5x+20)\sqrt{x^2+4x+5}-15\underbrace{\int\frac{dx}{\sqrt{x^2+4x+5}}}_{I_1}$$

$x^2+4x+5=x^2+2\cdot x\cdot 2+2^2-2^2+5=(x+2)^2+1$, $I_1=\displaystyle\int\frac{dx}{\sqrt{(x+2)^2+1}}=\ln|x+2+\sqrt{(x+2)^2+1}|+C$

$$I=(x^2-5x+20)\sqrt{x^2+4x+5}-15\ln|x+2+\sqrt{(x+2)^2+1}|+C$$

(2.) $I=\int\frac{3x+1}{\sqrt{2x^2-x+1}}\,dx \overset{Rj.}{=} a\sqrt{2x^2-x+1}+\lambda\int\frac{dx}{\sqrt{2x^2-x+1}} \quad \Big/ \frac{d}{dx}$

$$\frac{3x+1}{\sqrt{2x^2-x+1}} = a\cdot\frac{4x-1}{2\sqrt{2x^2-x+1}}+\lambda\cdot\frac{1}{\sqrt{2x^2-x+1}} \quad /\cdot 2\sqrt{2x^2-x+1}$$

$$6x+2 = a(4x-1)+2\lambda \Rightarrow \begin{array}{l} 4a=6 \\ 2\lambda-a=2 \end{array} \qquad a=\frac{6}{4}=\frac{3}{2}, \quad \lambda=\frac{7}{4}$$

$$I=\frac{3}{2}\sqrt{2x^2-x+1}+\frac{7}{4}\int\frac{dx}{\sqrt{2x^2-x+1}} = \frac{3}{2}\sqrt{2x^2-x+1}+\frac{7}{4}I_1 .$$

$$2x^2-x+1 = 2\left(x^2-\frac{1}{2}x+\frac{1}{2}\right) = 2\left(x^2-2\cdot x\cdot\frac{1}{4}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2+\frac{1}{2}\right) = 2\left[\left(x-\frac{1}{4}\right)^2+\frac{7}{16}\right]$$

$$I_1=\int\frac{dx}{\sqrt{2\left[\left(x-\frac{1}{4}\right)^2+\frac{7}{16}\right]}} = \frac{1}{\sqrt2}\int\frac{dx}{\sqrt{\left(x-\frac{1}{4}\right)^2+\frac{7}{16}}} = \left|\begin{array}{l} x-\frac{1}{4}=\frac{\sqrt7}{4}t \quad /\cdot 4 \\ dx=\frac{\sqrt7}{4}dt \\ 4x-1=\sqrt7\, t \end{array}\right| = \frac{1}{\sqrt2}\cdot\int\frac{\frac{\sqrt7}{4}dt}{\sqrt{\frac{7}{16}t^2-\frac{7}{16}}}$$

$$=\frac{1}{\sqrt2}\cdot\frac{\sqrt7}{4}\cdot\frac{4}{\sqrt7}\int\frac{dt}{\sqrt{t^2-1}} = \frac{1}{\sqrt2}\ln\left|t+\sqrt{t^2+1}\right|+c = \frac{1}{\sqrt2}\ln\left|\frac{4x-1}{\sqrt7}+\sqrt{\left(\frac{4x-1}{\sqrt7}\right)^2+1}\right|+C$$

$$I=\frac{3}{2}\sqrt{2x^2-x+1}+\frac{7}{4\sqrt2}\ln\left|\frac{4x-1}{\sqrt7}+\sqrt{\left(\frac{4x-1}{\sqrt7}\right)^2+1}\right|+C$$

Metodom Ostrogradskog rješavamo i integrale oblika

$$\int\sqrt{ax^2+bx+c}\,dx = \int\frac{ax^2+bx+c}{\sqrt{ax^2+bx+c}}\,dx .$$

(3.) $\int\sqrt{x^2+1}\,dx \overset{Rj.}{=} (ax+b)\sqrt{x^2+1}+\lambda\int\frac{dx}{\sqrt{x^2+1}} \quad \Big/\frac{d}{dx}$

$$\sqrt{x^2+1} = a\sqrt{x^2+1}+(ax+b)\frac{2x}{2\sqrt{x^2+1}}+\lambda\cdot\frac{1}{\sqrt{x^2+1}} \quad /\cdot\sqrt{x^2+1}$$

$$x^2+1 = a(x^2+1)+(ax^2+bx)+\lambda$$

x^2: $a+a=1$, $2a=1$, $a=\frac{1}{2}$ $\qquad$ x: $b=0$ $\qquad$ x^0: $a+\lambda=1$, $\lambda=1-\frac{1}{2}$, $\lambda=\frac{1}{2}$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|+C$$

(4.)✓ $\int\frac{2x^2-3x}{\sqrt{x^2-2x+5}}\,dx$ $\qquad$ (5.)✓ $\int\sqrt{x^2-2x-1}\,dx$

(6.)✓ $\int\frac{x^5}{\sqrt{1-x^2}}\,dx$ $\qquad$ Rj. $-\frac{8+4x^2+3x^4}{15}\sqrt{1-x^2}$

(7.) $\int x^4\sqrt{1-x^2}\,dx$ $\qquad$ uputa: $\int\frac{x^4(1-x^2)}{\sqrt{1-x^2}}\,dx$

$\int R(x,\sqrt[n]{ax+b})\,dx$, smjena $ax+b=t^n$

(1.) $$\int\frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}} = \left|\begin{array}{l} 2x-1=t^4 \\ 2dx=4t^3dt \quad /:2 \\ dx=2t^3dt \\ t=\sqrt[4]{2x-1} \end{array}\right| = 2\int\frac{t^3\,dt}{\sqrt{t^4}-\sqrt[4]{t^4}} = 2\int\frac{t^3}{\underbrace{t^2-t}_{t(t-1)}}dt$$

$$=2\int\frac{t^2}{t-1}dt = 2\int\frac{t^2-1+1}{t-1}dt = 2\int\frac{t^2-1}{t-1}dt+\int\frac{dt}{t-1} =$$

$$=2\int\frac{(t-1)(t+1)}{t-1}dt+2\int\frac{dt}{t-1} = 2\int(t-1)\,dt+2\int\frac{1}{t-1}dt = 2\cdot\frac{t^2}{2}-2t+2\ln|t-1|+c$$

$$=\sqrt[4]{(2x-1)^2}+2\sqrt[4]{(2x-1)}+2\ln\left|\sqrt[4]{2x-1}-1\right|+C = \sqrt{2x-1}+2\sqrt[4]{2x-1}+2\ln\left|\sqrt[4]{2x-1}-1\right|+C$$

(2.) $$\int\frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \left|\begin{array}{l} x=t^4 \\ dx=4t^3dt \\ t=\sqrt[4]{x} \end{array}\right| = \int\frac{\sqrt[3]{1+t}}{t^2}\cdot 4t^3\,dt =$$

$$=4\int t\sqrt[3]{1+t}\,dt = \left|\begin{array}{l} 1+t=u^3 \\ dt=3u^2du \\ u=\sqrt[3]{1+t} \\ t=u^3-1 \end{array}\right| = 4\int(u^3-1)\sqrt[3]{u^3}\cdot 3u^2\,du =$$

$$=12\int(u^6-u^3)\,du = 12\left(\frac{u^7}{7}-\frac{u^4}{4}\right)+C = \frac{12}{7}u^7-3u^4+c =$$

$$= \frac{12}{7}\sqrt[3]{(1+t)^7} - 3\sqrt[3]{(1+t)^4} + c = \frac{12}{7}\sqrt[3]{(1+\sqrt[4]{x})^7} - 3\sqrt[3]{(1+\sqrt[4]{x})^4} + C$$

3.) $\int \sqrt{\frac{x+1}{x-1}}\, dx$ ako stavim smjenu $\frac{x+1}{x-1} = t^2$ dobiću

$$x+1 = t^2(x-1)$$
$$x+1 = t^2x - t^2$$
$$x - t^2x = -t^2 - 1$$
$$(1-t^2)x = -t^2 - 1 \quad /(-1)$$
$$x = \frac{t^2+1}{t^2-1}$$

$$dx = d\left(\frac{t^2+1}{t^2-1}\right)$$
$$dx = \frac{2t(t^2-1) - (t^2+1)\cdot 2t}{(t^2-1)^2}\, dt$$
$$dx = \frac{2t^3 - 2t - 2t^3 - 2t}{(t^2-1)^2}\, dt$$

$$I = \int t \cdot \frac{-4t}{(t^2-1)^2}\, dt = -4\int t \cdot \frac{t}{(t^2-1)^2}\, dt = -4I_1 \qquad dx = \frac{-4t}{(t^2-1)^2}\, dt$$

$$I_1 = \int t \cdot \frac{t}{(t^2-1)^2}\, dt = \left| \begin{matrix} u = t \\ du = dt \end{matrix} \right. \qquad dv = \frac{t}{(t^2-1)^2}\, dt$$

$$V = \int \frac{t}{(t^2-1)^2}\, dt = \left| \begin{matrix} t^2 - 1 = z \\ 2t\,dt = dz \\ t\,dt = \frac{dz}{2} \end{matrix} \right| = \int \frac{\frac{1}{2}dz}{z^2} =$$

$$= \frac{1}{2}\int z^{-2}\, dz = \frac{1}{2}\cdot\frac{z^{-1}}{-1} = -\frac{1}{2}\cdot\frac{1}{z} = -\frac{1}{2(t^2-1)}$$

$$= -\frac{1}{2}\cdot\frac{t}{t^2-1} + \frac{1}{2}\int\frac{dt}{t^2-1} = -\frac{1}{2}\cdot\frac{t}{t^2-1} + \frac{1}{2}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c_1$$

$$I = -4\cdot\left(-\frac{1}{2}\cdot\frac{t}{t^2-1} + \frac{1}{4}\ln\left|\frac{t-1}{t+1}\right|\right) + c =$$

$$= 2\cdot\frac{\sqrt{\frac{x+1}{x-1}}}{\frac{x+1}{x-1} - 1} - 1\ln\left|\frac{\sqrt{\frac{x+1}{x-1}} - 1}{\sqrt{\frac{x+1}{x-1}} + 1}\right| + C$$

4.) $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$ Rj.: $6\sqrt[6]{x} - 3\sqrt[3]{x} + 2\sqrt{x} - 6\ln(1 + \sqrt[6]{x}) + C$

5.) $\int \frac{\sqrt{x+1} + 2}{(x+1)^2 - \sqrt{x+1}}\, dx$ Rj.: $\ln\left|\frac{(\sqrt{x+1} - 1)^2}{x+2+\sqrt{x+1}}\right| - \frac{2}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2\sqrt{x+1}+1}{\sqrt{3}} + C$

6.) $\int \frac{dx}{(2-x)\sqrt{1-x}}$ Rj. $-2\,\mathrm{arctg}\sqrt{1-x} + C$

integrali koji se mogu riješiti racionalisanjem

1.) $$I = \int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}}\, dx$$

Rj.

$$I = \int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} \cdot \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}}\, dx = \int \frac{x+1 - 2\cdot\sqrt{x+1}\cdot\sqrt{x-1} + x - 1}{x+1-(x-1)}\, dx =$$

$$= \frac{1}{2}\int (2x - 2\sqrt{x^2-1})\, dx = \int x\, dx - \int\sqrt{x^2-1}\, dx = \frac{x^2}{2} - I_1$$

$$I_1 = \int\sqrt{x^2-1}\, dx = (ax+b)\sqrt{x^2-1} + \lambda\int\frac{dx}{\sqrt{x^2-1}} \qquad /\frac{d}{dx}$$

$$\sqrt{x^2-1} = a\cdot\sqrt{x^2-1} + (ax+b)\frac{2x}{2\sqrt{x^2-1}} + \lambda\cdot\frac{1}{\sqrt{x^2-1}} \qquad /\sqrt{x^2-1}$$

$$x^2 - 1 = a(x^2-1) + (ax^2 + bx) + \lambda$$

x^2: $a + a = 1 \Rightarrow a = \frac{1}{2}$, x: $b = 0$, x^0: $-a + \lambda = -1$ $\lambda = -\frac{1}{2}$

$$I_1 = \int\sqrt{x^2-1}\, dx = \frac{1}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2-1}| + c$$

$$I = \frac{1}{2}x^2 - \frac{1}{2}\sqrt{x^2-1} + \frac{1}{2}\ln|x + \sqrt{x^2-1}| + c$$

2.) $$\int \frac{x}{\sqrt{x+2} + \sqrt{x+1}}\, dx$$

Rj.

$$\int \frac{x}{\sqrt{x+2} + \sqrt{x+1}} \cdot \frac{\sqrt{x+2} - \sqrt{x+1}}{\sqrt{x+2} - \sqrt{x+1}}\, dx = \int \frac{x\sqrt{x+2} - x\sqrt{x+1}}{x+2-(x+1)}\, dx =$$

$$= \int (x\sqrt{x+2} - x\sqrt{x+1})\, dx = \int x\sqrt{x+2}\, dx - \int x\sqrt{x+1}\, dx = I_1 - I_2$$

$$I_1=\int x\sqrt{x+2}\,dx=\left|\begin{array}{l}x+2=t^2\\ dx=2t\,dt\\ x=t^2-2\\ t=\sqrt{x+2}\end{array}\right|=\int(t^2-2)\,t\cdot 2t\,dt=2\int(t^4-2t^2)\,dt=2\cdot\frac{t^5}{5}-4\frac{t^3}{3}+C_1$$

$$=\frac{2}{5}\sqrt{(x+2)^5}-\frac{4}{3}\sqrt{(x+2)^3}+C_1$$

$$I_2=\int x\sqrt{x+1}\,dx=\left|\begin{array}{l}x+1=t^2\\ x=t^2-1\\ dx=2t\,dt\\ t=\sqrt{x+1}\end{array}\right|=\int(t^2-1)\cdot t\cdot 2t\,dt=2\int(t^4-t^2)\,dt=$$

$$=2\frac{t^5}{5}-2\cdot\frac{t^3}{3}+C_2=\frac{2}{5}\sqrt{(x+1)^5}-\frac{2}{3}\sqrt{(x+1)^3}+C_2$$

$$I=\frac{2}{5}\sqrt{(x+2)^5}-\frac{4}{3}\sqrt{(x+2)^3}-\frac{2}{5}\sqrt{(x+1)^5}+\frac{2}{3}\sqrt{(x+1)^3}+C$$

3° $\int\frac{dx}{x-\sqrt{x^2-1}}$. R: $\frac{1}{2}x^2+\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+C$

4° $\int\frac{dx}{\sqrt{x^2+1}-x}$

5° $\int\frac{\sqrt{x^2+2x+2}}{x}\,dx$

$$\int\frac{Mx+N}{(x-\alpha)^n\sqrt{ax^2+bx+c}}\,dx,\quad n\in\mathbb{N},\quad M,N,a,b,c\in\mathbb{R}$$

uvodimo smjenu $x-\alpha=\frac{1}{t}$

1° $$\int\frac{dx}{(x+1)\sqrt{x^2+x+1}}=\left|\begin{array}{l}x+1=\frac{1}{t}\\ dx=-\frac{1}{t^2}dt\\ t=\frac{1}{x+1}\end{array}\right|=\int\frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\underbrace{\left(\frac{1}{t}-1\right)^2+\frac{1}{t}-1+1}_{\frac{1}{t^2}-\frac{2}{t}+1+\frac{1}{t}}}}=-\int\frac{dt}{t\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}}$$

$$=-\int\frac{dt}{\sqrt{t^2-t+1}}=\left|\begin{array}{l}t^2-t+1=\\ =t^2-2\cdot t\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1\\ =\left(t-\frac{1}{2}\right)^2+\frac{3}{4}\end{array}\right|=-\int\frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}}=\left|\begin{array}{l}t-\frac{1}{2}=\frac{\sqrt{3}}{2}z\\ dt=\frac{\sqrt{3}}{2}dz\\ \sqrt{3}z=2t-1\end{array}\right|$$

$$=-\frac{\sqrt{3}}{2}\int\frac{dz}{\sqrt{\frac{3}{4}z^2+\frac{3}{4}}}=-\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}}\int\frac{dz}{\sqrt{z^2+1}}=-\ln|z+\sqrt{z^2+1}|+C=$$

$$=-\ln\left|\frac{2t-1}{\sqrt{3}}+\sqrt{\left(\frac{2t-1}{\sqrt{3}}\right)^2+1}\right|+C=-\ln\left|\frac{\frac{2}{x+1}-1}{\sqrt{3}}+\sqrt{\frac{\left(\frac{2}{x+1}-1\right)^2}{3}+1}\right|+C$$

2° $$I=\int\frac{dx}{(x-1)^3\sqrt{x^2+3x+1}}=\left|\begin{array}{l}x-1=\frac{1}{t}\\ dx=-\frac{1}{t^2}dt\\ x=\frac{1}{t}+1\\ t=\frac{1}{x-1}\end{array}\right|=\int\frac{-\frac{1}{t^2}dt}{\frac{1}{t^3}\sqrt{\underbrace{\left(\frac{1}{t}+1\right)^2+3\left(\frac{1}{t}+1\right)+1}_{\frac{1}{t^2}+\frac{2}{t}+1+\frac{3}{t}+3+1}}}=-\int\frac{t\,dt}{\sqrt{\frac{1}{t^2}+\frac{5}{t}+[illegible]}}$$

$$=-\int\frac{t\,dt}{\sqrt{\frac{1+5t+5t^2}{t^2}}}=-\int\frac{t^2}{\sqrt{5t^2+5t+1}}\,dt=(at+b)\sqrt{5t^2+5t+1}+\lambda\int\frac{dt}{\sqrt{5t^2+5t+1}}\quad\Big/\frac{d}{dt}$$

$$\frac{-t^2}{\sqrt{5t^2+5t+1}}=a\sqrt{5t^2+5t+1}+(at+b)\frac{10t+5}{2\sqrt{5t^2+5t+1}}+\lambda\cdot\frac{1}{\sqrt{5t^2+5t+1}}\quad\Big/\cdot 2\sqrt{5t^2+5t+1}$$

$$-2t^2=2a\cdot(5t^2+5t+1)+a(10t^2+5t)+b(10t+5)+2\lambda$$

t^2: $10a+10a=-2$, $a=-\frac{1}{10}$

t: $10a+5a+10b=0$, $15a+10b=0$, $10b=\frac{15}{10}$

$10b=\frac{3}{2}$, $b=\frac{3}{20}$

t^0: $2a+5b+2\lambda=0$

$-\frac{2}{10}+\frac{15}{20}=-2\lambda$, $-2\lambda=\frac{11}{20}$, $\lambda=-\frac{11}{40}$

$$I=\left(-\frac{1}{10}t+\frac{3}{20}\right)\sqrt{5t^2+5t+1}-\frac{11}{40}\int\frac{dt}{\sqrt{5t^2+5t+1}}\qquad\text{SAMI ZAVRŠITI}$$

3° $\int\frac{dx}{x^2\sqrt{x^2-x+1}}$ 4° $\int\frac{(3x+2)\,dx}{(x+1)\sqrt{x^2+3x+3}}$ 5° $\int\frac{dx}{x^3\sqrt{x^2+1}}$

integral binomnog diferencijala

$$\int x^m(a+bx^n)^p\,dx\qquad(a,b\in\mathbb{R};\ m,n,p\in\mathbb{Q})$$

Integracija je moguća ako

1° $p\in\mathbb{Z}$, uvodimo smjenu $x=t^s$, $s=NZS(m_2,n_2)$ — nazivnika od m i n, $m=\frac{m_1}{m_2}$, $n=\frac{n_1}{n_2}$

2° $\frac{m+1}{n}\in\mathbb{Z}$, uvodimo smjenu $a+bx^n=t^{p_2}$, $p=\frac{p_1}{p_2}$

3° $\frac{m+1}{n}+p\in\mathbb{Z}$, uvodimo smjenu $ax^{-n}+b=t^{p_2}$, p_2 nazivnik od p

integral binomnog diferencijala

$$\int x^m (a+bx^n)^p\,dx \quad (a,b\in\mathbb{R};\ m,n,p\in\mathbb{Q})$$

Integracija je moguća ako

1° $p\in\mathbb{Z}$, uvodimo smjenu $x=t^s$, $s=NZS(m_2,n_2)$, $m=\frac{m_1}{m_2}$, $n=\frac{n_1}{n_2}$
nazivnika od m i n

2° $\frac{m+1}{n}\in\mathbb{Z}$, uvodimo smjenu $a+bx^n=t^{p_2}$, $p=\frac{p_1}{p_2}$

3° $\frac{m+1}{n}+p\in\mathbb{Z}$, uvodimo smjenu $ax^{-n}+b=t^{p_2}$, p_2 nazivnik od p

1₀ $$\int\frac{dx}{x^2(\sqrt{1+x^2})^3}=\int x^{-2}(1+x^2)^{-\frac32}dx=\left|\begin{array}{l} m=-2,\ n=2,\ p=-\frac32\\ p\notin\mathbb{Z},\ \text{nije } 1°\\ \frac{m+1}{n}=\frac{-2+1}{2}=-\frac12\notin\mathbb{Z},\ \text{nije } 2°\\ \frac{m+1}{n}+p=-\frac12-\frac32=-2\in\mathbb{Z},\ 3°\end{array}\right.$$

smjena: $x^{-2}+1=t^2$

$x^{-2}=t^2-1$

$x^2=(t^2-1)^{-1}$

$x=(t^2-1)^{-\frac12}$

$dx=-\frac12(t^2-1)^{-\frac32}\cdot 2t\,dt$

$dx=-t(t^2-1)^{-\frac32}dt$

$1+x^2=1+(t^2-1)^{-1}$

$$=\int(t^2-1)\left(1+\frac{1}{t^2-1}\right)^{-\frac32}\cdot(-t)(t^2-1)^{-\frac32}dt$$

$$=\int(t^2-1)\cdot\frac{t^{-3}}{(t^2-1)^{-\frac32}}(-t)\cdot(t^2-1)^{-\frac32}dt$$

$$=\int(-1+t^{-2})\,dt=-t-\frac1t+c=\frac{-x^{-2}-2}{\sqrt{x^{-2}+1}}+C$$

Eulerove smjene $\int R(x,\sqrt{ax^2+bx+c})\,dx$ R-racionalna f-ja

1° za $a>0$ uzimamo smjenu $\sqrt{ax^2+bx+c}=\pm\sqrt{a}\,x+t$

2° za $c>0$ uzimamo smjenu $\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$

3° za $b^2-4ac>0$ uzimamo smjenu $\sqrt{a(x-x_1)(x-x_2)}=t(x-x_1)$

$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}$

1₀ $$I=\int\frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}$$ Rj. 3°
smjena $\sqrt{(x-1)(x+2)}=t(x+1)$

Integral binomnog diferencijala

$$\int x^m(a+bx^n)^p\,dx \quad (m,n,p\in\mathbb{Q})$$

Integracija je moguća:

1° $p\in\mathbb{Z}$

smjena: $x=t^s$

s = NZS nazivnika od m i n

2° $\frac{m+1}{n}\in\mathbb{Z}$

$a+bx^n=t^s$ – smjena

s – nazivnik od p

3° $\frac{m+1}{n}+p\in\mathbb{Z}$

$ax^{-n}+b=t^s$

s – nazivnik od p

$\sqrt[n]{x^m}=x^{\frac{m}{n}}$

1. $$\int x^{-\frac34}\cdot\left(1+x^{\frac16}\right)^{-1}dx=\left|\begin{array}{l} m=-\frac34,\ n=\frac16,\ p=-1\\ x=t^{12}\Rightarrow dx=12t^{11}dt\end{array}\right|=$$

$$=\int(t^{12})^{-\frac34}\cdot\left(1+(t^{12})^{\frac16}\right)^{-1}\cdot12t^{11}dt=$$

$$=12\int t^{-9}\cdot(1+t^2)^{-1}\cdot t^{11}dt=12\int\frac{t^2+1-1}{1+t^2}dt=$$

$$=12\int\left(1-\frac{1}{t^2+1}\right)dt=12\cdot\int dt-\int\frac{1}{t^2+1}=12t-\mathrm{arctg}\,t+C$$

$$=12\sqrt[12]{x}-\mathrm{arctg}\sqrt[12]{x}+C$$

2. $$\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = \int x^{-\frac{2}{3}}\cdot\left(1+x^{\frac{1}{3}}\right)^{\frac{1}{2}}dx =$$

$m = \frac{-2}{3}$, $n = \frac{1}{3}$, $p = \frac{1}{2}$

$\frac{m+1}{n} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$

2° slučaj

$1+x^{\frac{1}{3}} = t^2$

$\frac{1}{3}x^{\frac{1}{3}-1}dx = 2t\,dt \quad /\cdot 3$

$x^{-\frac{2}{3}}dx = 6t\,dt$

$$= 6\int (t^2)^{\frac{1}{2}}\cdot t\,dt = 6\int t^2\,dt = 6\cdot\frac{t^3}{3} + C = 2t^3 + C$$

$$= 2\cdot\left(\sqrt{1+\sqrt[3]{x}}\right)^3 + C$$

3. $$\int \frac{dx}{x^2\left(\sqrt{1+x^2}\right)^3} = \int x^{-2}\left(1+x^2\right)^{-\frac{3}{2}}dx =$$

$m=-2$, $n=2$, $p=-\frac{3}{2}$

$\frac{m+1}{n}+p = -\frac{1}{2}-\frac{3}{2} = -2$

3° slučaj

smjena: $x^{-2}+1 = t^2$

$x^{-2} = t^2-1$

$x^2 = (t^2-1)^{-1}$

$x = (t^2-1)^{-\frac{1}{2}}$

$dx = -\frac{1}{2}\cdot(t^2-1)^{-\frac{3}{2}}\cdot 2t\,dt$

$dx = -t\cdot(t^2-1)^{-\frac{3}{2}}dt$

$$= \int (t^2-1)\cdot\left(1+\frac{1}{t^2-1}\right)^{-\frac{3}{2}}\cdot(-t)\cdot(t^2-1)^{-\frac{3}{2}}dt =$$

$$= \int (t^2-1)\cdot\left(\frac{t^2-1+1}{t^2-1}\right)^{-\frac{3}{2}}\cdot(-t)\cdot(t^2-1)^{-\frac{3}{2}}dt =$$

$$\int (t^2-1)\cdot\frac{t^{-3}}{\cancel{(t^2-1)^{-\frac{3}{2}}}}\cdot(-t)\cdot\cancel{(t^2-1)^{-\frac{3}{2}}}\,dt$$

$$= \int(-1+t^{-2})\,dt = -t-\frac{1}{t}+C = \frac{-t^2-1}{t}+C =$$

$$= \frac{-x^{-2}-1-1}{\sqrt{x^{-2}+1}} + C = \frac{-x^{-2}-2}{\sqrt{x^{-2}+1}} + C$$

za vježbu:

4. $\int \frac{dx}{\sqrt[3]{(1+x^2)^2}}$ (uspješno)

5. $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx$ – I tip

6. $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$ – II tip

7. $\int \frac{dx}{x^3\cdot\sqrt[3]{2-x^3}}$ – III tip

Eulerove smjene

$$I = \int R\left(x, \sqrt{ax^2+bx+c}\right)dx$$

R – racionalna funkcija

1° $\sqrt{ax^2+bx+c} = \pm\sqrt{a}\cdot x + t$ (ako je $a>0$)

2° $\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}$ (ako je $c>0$)

3° $\sqrt{ax^2+bx+c} = t\cdot(x-x_1)$ (ako je $ax^2+bx+c = a(x-x_1)(x-x_2)$, $x_1, x_2 \in \mathbb{R}$)

1. $I = \int \frac{dx}{x+\sqrt{x^2+x+1}}$

smjena: $\sqrt{x^2+x+1} = -x+t \quad /^2$

$$\cancel{x^2}+x+1 = \cancel{x^2}-2xt+t^2$$

$$x+2xt = t^2-1$$

$$x\cdot(1+2t) = t^2-1$$

$$x = \frac{t^2-1}{1+2t}$$

$$dx = \frac{2t\cdot(1+2t)-2\cdot(t^2-1)}{(1+2t)^2}$$

$$dx = \frac{2t+4t^2-2t^2+2}{(1+2t)^2}dt$$

$$dx = \frac{2t^2+2t+2}{(1+2t)^2}dt$$

$$I = \int \frac{\frac{2t^2+2t+2}{(1+2t)^2}}{t}dt =$$

$$= \int \frac{2t^2+2t+2}{t\cdot(1+2t)^2}dt$$

$$I = \frac{dx}{1+\sqrt{1-2x-x^2}}$$

$c = 1$

$$xt-1 = \sqrt{1-2x-x^2} \quad /^2$$

$$x^2t^2-2xt+\cancel{1} = \cancel{1}-2x-x^2 \quad /:x$$

$$xt^2-2t = -2-x$$

$$xt^2+x = 2t-2$$

$$x\cdot(t^2+1) = 2t-2$$

$$x = \frac{2t-2}{t^2+1}$$

$$dx = \frac{2(t^2+1)-2t(2t-2)}{(t^2+1)^2}dt$$

$$dx = \frac{2t^2+2-4t^2+4t}{(t^2+1)^2}dt$$

$$dx = \frac{-2t^2+4t+2}{(t^2+1)^2}dt$$

$$I = \int \frac{\frac{-2t^2+4t+2}{(t^2+1)^2}}{\frac{2t-2}{t^2+1}\cdot t}dt = \int \frac{\cancel{2}\cdot(-t^2+2t+1)dt}{\cancel{2}\cdot(t-1)\cdot t\cdot(t^2+1)}$$

$$\frac{-t^2+2t+1}{t\cdot(t-1)(t^2+1)} = \frac{A}{t}+\frac{B}{t-1}+\frac{Ct+D}{t^2+1}$$ ... A, B, C, D za vj.

$$A = -1, \quad B = 1, \quad C = 0, \quad D = 2$$

$$= -\int\frac{1}{t}dt + \int\frac{1}{t-1}dt + \int\frac{2dt}{t^2+1} =$$

$$= -\ln|t| + \ln|t-1| + 2\,\mathrm{arctg}\, t + C$$

$$= \ln\left|\frac{t-1}{t}\right| + 2\,\mathrm{arctg}\, t + C$$

$$t = \frac{1+\sqrt{1-2x-x^2}}{x}$$

3. $I=\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx$

$$x^2+3x+2=(x+1)\cdot(x+2)$$

$$\sqrt{x^2+3x+2}=t\,(x+1)\ \Big/^2$$

$$x^2+3x+2=t^2(x+1)^2$$

$$(x+1)(x+2)=t^2(x+1)^2\ \Big/:(x+1)$$

$$x+2=t^2\cdot(x+1)$$

$$x+2=t^2x+t^2$$

$$x-t^2x=t^2-2$$

$$x(1-t^2)=t^2-2$$

$$x=\frac{t^2-2}{1-t^2}$$

$$dx=\frac{2t\cdot(1-t^2)+2t\cdot(t^2-2)}{(1-t^2)^2}\,dt$$

$$dx=\frac{2t-2t^3+2t^3-4t}{(1-t^2)^2}\,dt$$

$$dx=\frac{-2t}{(1-t^2)^2}\,dt$$

$$I=\int\frac{\frac{t^2-2}{1-t^2}+\frac{t}{1-t^2}}{\frac{t^2-2}{1-t^2}-\frac{t}{1-t^2}}\cdot\frac{-2t}{(1-t^2)^2}\,dt=$$

$$\left[\sqrt{x^2+3x+2}=t\cdot\left(\frac{t^2-2}{1-t^2}+1\right)=t\cdot\frac{t^2-2+1-t^2}{1-t^2}=\frac{-t}{1-t^2}\right]$$

$$=\int\frac{(t^2+t-2)\cdot(-2t)}{(t^2-t-2)(1-t^2)^2}\,dt$$

$$=-2\int\frac{\cancel{(t-1)}(t+2)\,t}{(t+1)\cdot(t-2)\cdot(t-1)^{\cancel{2}}(t+1)^2}\,dt$$

$$-2\int\frac{(t^2+2t)}{(t-1)\cdot(t-2)\cdot(t+1)^3}\,dt$$

$$=\frac{-2t^2-4t}{(t-1)\cdot(t-2)\,(t+1)^3}=\frac{A}{t-1}+\frac{B}{t-2}+\frac{C}{t+1}+\frac{D}{(t+1)^2}+\frac{E}{(t+1)^3}$$

Za vježbu:

a) $\int\frac{1-\sqrt{x^2+x+1}}{x\cdot\sqrt{x^2+x+1}}\,dx$ (racionalisati

b) $\int x\cdot\sqrt{x^2-2x+2}\,dx$

d) $\int\frac{x^2\,dx}{(4-2x+x^2)\cdot\sqrt{2+2x-x^2}}$

c) $\int\frac{dx}{\left[1+\sqrt{x\cdot(1+x)}\right]^2}$

9. Integracija nekih transcedentnih (nealgebarskih) f-ja

U ovoj lekciji ćemo posmatrati sljedeće tipove integrala (R predstavlja racionalnu f-ju)

I. $\int R(\sin x, \cos x)\,dx$ – uvodimo smjenu $z = tg\frac{x}{2}$,

pri čemu je $\sin x = \frac{2z}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$, $dx = \frac{2dz}{1+z^2}$

II. $\int R(tg\,x)\,dx$ – uvodimo smjenu $tg\,x = z$,

pri čemu je $x = arc\,tg\,z$, $dx = \frac{dz}{1+z^2}$

III. $\int R(e^x)\,dx$ – uvodimo smjenu $e^x = z$,

pri čemu je $x = \ln z$, $dx = \frac{dz}{z}$.

$$z = tg\frac{x}{2} \Rightarrow \frac{x}{2} = arc\,tg\,z \Rightarrow x = 2\,arc\,tg\,z$$

$$dx = \frac{2\,dz}{1+z^2},\quad \sin x = \sin 2\cdot\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\ \Big/{:\cos^2\tfrac{x}{2}} =$$

$$= \frac{2\,tg\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{2z}{1+z^2}\quad \text{tj. } \sin x = \frac{2z}{1+z^2}$$

$$\cos x = \cos 2\cdot\frac{x}{2} = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\ \Big/{:\cos^2\tfrac{x}{2}} = \frac{1-z^2}{1+z^2}$$

Odrediti integrale

a) $\int \frac{dx}{2\sin x - \cos x}$; b) $\int \frac{dx}{5+4\cos ax}$;

c) $\int \frac{tg\,x\,dx}{1-ctg^2 x}$; d) $\int \frac{e^{3x}dx}{e^{2x}+1}$.

Rj.

a)
$$\int \frac{dx}{2\sin x - \cos x} = \left| \begin{array}{ll} tg\frac{x}{2} = z & \sin x = \frac{2z}{1+z^2} \\ dx = \frac{2dz}{1+z^2} & \cos x = \frac{1-z^2}{1+z^2} \end{array} \right| =$$

$$= \int \frac{\frac{2dz}{1+z^2}}{\frac{4z}{1+z^2} - \frac{1-z^2}{1+z^2}} = \int \frac{2\,dz}{z^2+4z-1} \overset{(*)}{=} 2\int \frac{d(z+2)}{(z+2)^2-5} =$$

$$z^2+4z-1 = z^2+2\cdot z\cdot 2+4-4-1 = (z+2)^2-5$$
$$d(z+2) = dz \quad \ldots(*)$$

$$= 2\cdot\frac{1}{2\sqrt{5}}\ln\left|\frac{z+2-\sqrt{5}}{z+2+\sqrt{5}}\right| + C = \frac{1}{\sqrt{5}}\ln\left|\frac{tg\frac{x}{2}+2-\sqrt{5}}{tg\frac{x}{2}+2+\sqrt{5}}\right| + C$$

b)
$$\int \frac{dx}{5+4\cos ax} = \left| \begin{array}{ll} tg\frac{ax}{2} = z \Rightarrow \frac{ax}{2} = arc\,tg\,z & \\ dx = \frac{2\,dz}{a(1+z^2)}, & \cos ax = \frac{1-z^2}{1+z^2} \end{array} \right| =$$

$$= \int \frac{\frac{2\,dz}{a(1+z^2)}}{5+\frac{4(1-z^2)}{1+z^2}} = \frac{2}{a}\int \frac{dz}{5+5z^2+4-4z^2} = \frac{2}{a}\int \frac{dz}{z^2+9} =$$

$$= \frac{2}{a}\cdot\frac{1}{3}\,\mathrm{arctg}\,\frac{z}{3} + c = \frac{2}{3a}\,\mathrm{ar}\,\mathrm{tg}\left(\frac{1}{3}\,\mathrm{tg}\,\frac{ax}{2}\right) + c$$

c) $$\int\frac{\mathrm{tg}\,x\,dx}{1-\mathrm{ctg}^2 x} = \left|\begin{array}{ll} \mathrm{tg}\,x = z, & x = \mathrm{arctg}\,z \\ dx = \dfrac{dz}{1+z^2} & \mathrm{ctg}^3 x = \left(\dfrac{1}{\mathrm{tg}\,x}\right)^2 = \dfrac{1}{z^2}\end{array}\right|$$

$$= \int\frac{\dfrac{z\,dz}{1+z^2}}{1-\dfrac{1}{z^2}} = \int\frac{\dfrac{z\,dz}{1+z^2}}{\dfrac{z^2-1}{z^2}} = \int\frac{z^3\,dz}{z^4-1} =$$

$$= \int\frac{\frac{1}{4}\,d(z^4-1)}{z^4-1} = \frac{1}{4}\ln|z^4-1| + c = \frac{1}{4}\ln|\mathrm{tg}^4 x - 1| + c$$

d) $$\int\frac{e^{3x}\,dx}{e^{2x}+1} = \left|\begin{array}{c} e^x = z \\ e^x dx = dz \\ dx = \dfrac{dz}{z}\end{array}\right| = \int\frac{z^3\cdot\frac{dz}{z}}{z^2+1} =$$

$$= \int\frac{z^2\,dz^{+1-1}}{z^2+1} = \int\left(1-\frac{1}{z^2+1}\right)dz = \int dz - \int\frac{dz}{z^2+1}$$

$$= z - \mathrm{arctg}\,z + c = e^x - \mathrm{arctg}\,e^x + c$$

Zadaci za vježbu

1. $\displaystyle\int\frac{\cos x\,dx}{1+\cos x}$

2. $\displaystyle\int\frac{dx}{\sin kx}$

3. $\displaystyle\int\frac{dx}{\sin^3 x}$

4. $\displaystyle\int\frac{dx}{4\cos x + 3\sin x}$

5. $\displaystyle\int \mathrm{tg}^5\,3x\,dx$

6. $\displaystyle\int\frac{dx}{1+\mathrm{tg}\,x}$

7. $\displaystyle\int\frac{e^{2t}-2e^t}{1+e^{2t}}\,dt$

8. $\displaystyle\int\frac{e^x-1}{e^x+1}\,dx$

9.* $\displaystyle\int\frac{1+\mathrm{tg}\,x}{\sin 2x}\,dx$

10.* $\displaystyle\int\frac{e^{2x}\,dx}{(2+e^x+e^{-x})^2}$

Rjesenja

1. $x - \mathrm{tg}\,\frac{x}{2}$ 2. $\frac{1}{k}\left|\mathrm{tg}\,\frac{kx}{2}\right|$ 3. $\frac{1}{2}\left(\ln\left|\mathrm{tg}\,\frac{x}{2}\right| - \frac{\cos x}{\sin^2 x}\right)$ 4. $\frac{1}{5}\ln\left|\frac{1+2\,\mathrm{tg}\,\frac{x}{2}}{2-\mathrm{tg}\,\frac{x}{2}}\right|$ 5. $\frac{1}{12}\,\mathrm{tg}^4\,3x - \frac{1}{6}\,\mathrm{tg}^2\,3x - \frac{1}{3}\ln|\cos 3x|$ 6. $\frac{1}{2}(x+\ln|\sin x+\cos x|)$

7. $\frac{1}{2}\ln(e^{2t}+1) - 2\,\mathrm{arctg}\,e^t$ 8. $2\ln(e^x+1) - x$

9. $\frac{1}{2}(\mathrm{tg}\,x + \ln|\mathrm{tg}\,x|)$ 10. $\ln(e^x+1) + \frac{18e^{2x}+27e^x+11}{6(e^x+1)^3}$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija nekih nealgebarskih funkcija)

$\int R(\sin x, \cos x)\,dx$, R - racionalna f-ja

koristimo smjenu $tg\frac{x}{2} = t \Rightarrow$

$$\Rightarrow dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$tg\frac{x}{2} = t$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2} \;:\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2} \;:\cos^2\frac{x}{2}} = \frac{2tg\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2} \;:\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2} \;:\cos^2\frac{x}{2}} = \frac{1-tg^2\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{1-t^2}{1+t^2}$$

$$tg\frac{x}{2} = t \Rightarrow \frac{x}{2} = arctg\,t \Rightarrow x = 2arctg\,t \Rightarrow dx = \frac{2dt}{1+t^2}$$

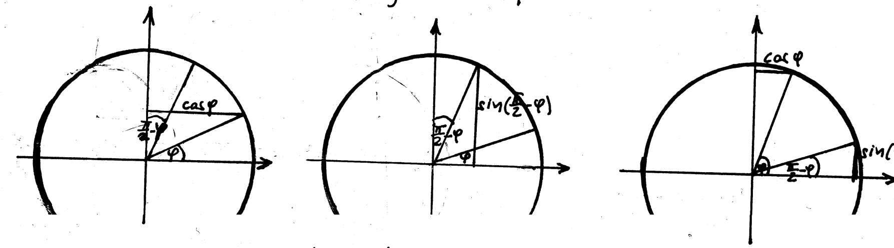
1. $$\int\frac{dx}{\sin x} = \left|\begin{matrix} tg\frac{x}{2} = t \\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2}\end{matrix}\right| = \int\frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int\frac{dt}{t} = \ln|t| + C = \ln\left|tg\frac{x}{2}\right| + C$$

2. $$\int\frac{dx}{\cos x} = \left|\begin{matrix} \sin x = \cos(\frac{\pi}{2}-x) \\ \cos x = \sin(\frac{\pi}{2}-x) \\ \text{OBJASNI} \\ \text{OVO}\end{matrix}\right| = \int\frac{dx}{\sin(\frac{\pi}{2}-x)} = \left|\begin{matrix} \frac{\pi}{2}-x = t \\ -dx = dt \\ dx = -dt\end{matrix}\right| =$$

$$= -\int\frac{dt}{\sin t} \;\overset{1.zad.}{=}\; -\ln\left|tg\frac{t}{2}\right| + C = -\ln\left|tg\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C =$$

$$= \ln\left|tg\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|^{-1} = \ln\left|ctg\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C = \left|\begin{matrix} ctg\,x = \frac{\cos x}{\sin x} = \\ = \frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)} = tg\left(\frac{\pi}{2}-x\right)\end{matrix}\right|$$

$$= \ln\left|tg\,\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C = \ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$$



3. $$\int\frac{dx}{5-4\sin x+3\cos x} = \left|\begin{matrix} tg\frac{x}{2} = t \\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2}\end{matrix}\right| = \int\frac{\frac{2dt}{1+t^2}}{5-4\cdot\frac{2t}{1+t^2}+3\cdot\frac{1-t^2}{1+t^2}} =$$

$$= 2\int\frac{\frac{dt}{\cancel{1+t^2}}}{\frac{5+5t^2-8t+3-3t^2}{\cancel{1+t^2}}} = 2\int\frac{dt}{2t^2-8t+8} = \int\frac{dt}{t^2-4t+4}$$

$$= \int\frac{dt}{(t-2)^2} = \left|\begin{matrix} t-2 = z \\ dt = dz\end{matrix}\right| = \int\frac{dz}{z^2} = -\frac{1}{z} + C = -\frac{1}{t-2} + C = -\frac{1}{tg\frac{x}{2}-2} + C$$

4. ˇ $$I = \int\frac{\cos x+2\sin x}{4\cos x+3\sin x}\,dx = \left|\begin{matrix} tg\frac{x}{2} = t & \cos x = \frac{1-t^2}{1+t^2} \\ dx = \frac{2dt}{1+t^2} & \sin x = \frac{2t}{1+t^2}\end{matrix}\right| =$$

$$= \int\frac{\frac{1-t^2}{1+t^2}+2\frac{2t}{1+t^2}}{4\cdot\frac{1-t^2}{1+t^2}+3\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2} = \int\frac{\frac{1-t^2+4t}{1+t^2}}{\frac{4-4t^2+6t}{1+t^2}}\cdot\frac{2dt}{1+t^2} =$$

$$= \int\frac{-t^2+4t+1}{(-2t^2+3t+2)(1+t^2)}\,dt = \ldots$$

5. ˇ $$\int\frac{dx}{8-4\sin x+7\cos x}$$

6. ˇ $$\int\frac{\cos x+\sin x}{\cos x-2\sin x}\,dx$$

$$\int R(\sin^2 x, \sin x\cos x, \cos^2 x)\,dx$$ R- racionalna f-ja

ili $$\int R(\operatorname{tg} x)\,dx$$

uvodimo smjenu $\operatorname{tg} x = t \Rightarrow$

$$\Rightarrow dx = \frac{dt}{1+t^2}, \quad \sin^2 x = \frac{t^2}{1+t^2}, \quad \cos^2 x = \frac{1}{1+t^2},$$

$$\sin x \cos x = \frac{t}{1+t^2}$$

$$\operatorname{tg} x = t \Rightarrow x = \operatorname{arc\,tg} t \Rightarrow dx = \frac{dt}{1+t^2}$$

$$\sin^2 x = \frac{\sin^2 x \quad :\cos^2 x}{\sin^2 x + \cos^2 x \quad :\cos^2 x} = \frac{\operatorname{tg}^2 x}{\operatorname{tg}^2 x + 1} = \frac{t^2}{1+t^2}$$

$$\cos^2 x = \frac{\cos^2 x \quad :\cos^2 x}{\sin^2 x + \cos^2 x \quad :\cos^2 x} = \frac{1}{\operatorname{tg}^2 x + 1} = \frac{1}{1+t^2}$$

$$\sin x \cos x = \frac{\sin x \cos x \quad :\cos^2 x}{\sin^2 x + \cos^2 x \quad :\cos^2 x} = \frac{\operatorname{tg} x}{\operatorname{tg}^2 x + 1} = \frac{t}{1+t^2}$$

1. $$\int \frac{dx}{\cos^4 x} = \left| \begin{array}{ll} \operatorname{tg} x = t & \cos^2 x = \frac{1}{1+t^2} \\ dx = \frac{dt}{1+t^2} & \end{array} \right| = \int \frac{\frac{dt}{1+t^2}}{\left(\frac{1}{1+t^2}\right)^2} = \int \frac{(1+t^2)^{\cancel{2}}}{\cancel{1+t^2}}\,dt =$$

$$= \int (1+t^2)\,dt = \int dt + \int t^2\,dt = t + \frac{t^3}{3} + C = \operatorname{tg} x + \frac{1}{3}\operatorname{tg}^3 x + C$$

2. $$\int \frac{dx}{\sin^2 x - 4\sin x\cos x + 5\cos^2 x} = \left| \begin{array}{lll} \operatorname{tg} x = t & dx = \frac{dt}{1+t^2} & \sin x\cos x = \\ \sin^2 x = \frac{t^2}{1+t^2} & \cos^2 x = \frac{1}{1+t^2} & = \frac{t}{1+t^2} \end{array} \right|$$

$$= \int \frac{\frac{dt}{1+t^2}}{\frac{t^2}{1+t^2} - 4\cdot\frac{t}{1+t^2} + 5\cdot\frac{1}{1+t^2}} = \int \frac{dt}{t^2 - 4t + 5} = \int \frac{dt}{(t-2)^2 + 1} = \operatorname{arctg}(t-2) + C$$

$$= \operatorname{arc\,tg}(\operatorname{tg} x - 2) + C$$

3. $$\int \operatorname{tg}^3 x\,dx = \left| \begin{array}{l} \operatorname{tg} x = t \\ dx = \frac{dt}{1+t^2} \end{array} \right| = \int t^3 \cdot \frac{dt}{1+t^2} = \int \frac{t^3 + t - t}{1+t^2}\,dt = \int \frac{t + t^3}{1+t^2}\,dt -$$

$$- \int \frac{t}{1+t^2}\,dt = \int \frac{\cancel{t(1+t^2)}}{\cancel{(1+t^2)}}\,dt - \frac{1}{2}\int \frac{2t\,dt}{1+t^2} = \left| \begin{array}{l} t^2 = s \\ 2t\,dt = ds \end{array} \right| =$$

$$= \int t\,dt - \frac{1}{2}\int \frac{ds}{1+s} = \frac{t^2}{2} - \frac{1}{2}\ln|1+s| + C = \frac{1}{2}t^2 - \frac{1}{2}\ln|t^2+1| + C =$$

$$= \frac{1}{2}\operatorname{tg}^2 x - \frac{1}{2}\ln|\operatorname{tg}^2 + 1| + C = \frac{1}{2}\operatorname{tg}^2 x - \frac{1}{2}\ln\left|\frac{\sin^2 x}{\cos^2 x} + 1\right| + C = \frac{1}{2}\operatorname{tg}^2 x - \frac{1}{2}\ln\left|\frac{1}{\cos^2 x}\right| + C$$

$$= \frac{1}{2}\operatorname{tg}^2 x + \ln\left|\frac{1}{\cos^2 x}\right|^{-\frac{1}{2}} + C = \frac{1}{2}\operatorname{tg}^2 x + \ln|\cos^2 x|^{\frac{1}{2}} + C = \frac{1}{2}\operatorname{tg}^2 x + \ln|\cos x| + C$$

4. ✓ $$\int \frac{\cos x + 2\sin x \quad :\cos x}{4\cos x + 3\sin x \quad :\cos x}\,dx = \int \frac{1 + 2\operatorname{tg} x}{4 + 3\operatorname{tg} x}\,dx = \left| \begin{array}{l} \operatorname{tg} x = t \\ dx = \frac{dt}{1+t^2} \end{array} \right| = \int \frac{1+2t}{4+3t}\cdot\frac{dt}{1+t^2}$$

$$\frac{1+2t}{(4+3t)(1+t^2)} = \frac{a}{4+3t} + \frac{bt + c}{1+t^2}$$

...

5. ✓ $$\int \frac{dx}{\sin^4 x}$$

6. ✓ $$\int \frac{dx}{3\cos^2 x + 4\sin^2 x}$$

7. ✓ $$\int \frac{\operatorname{tg} x}{\operatorname{tg}^2 x - 2\operatorname{tg} x - 3}\,dx \rightarrow$$ Rj. $-\frac{1}{10}x + \frac{3}{40}\ln|\operatorname{tg} x - 3| + \frac{1}{8}\ln|\operatorname{tg} x + 1| + \frac{1}{5}\ln|\cos x| + C$

Izračunati integral $I=\int \frac{dx}{3\cos^2 x+4\sin^2 x}$.

Rj. $\operatorname{tg} x = t$ $\quad \sin^2 x = \frac{\sin^2 x}{\sin^2 x+\cos^2 x}\begin{matrix}:\cos^2 x\\ :\cos^2 x\end{matrix} = \frac{\operatorname{tg}^2 x}{\operatorname{tg}^2 x+1} = \frac{t^2}{1+t^2}$

$x = \operatorname{arc\,tg} t$

$dx = \frac{dt}{1+t^2}$ $\quad \cos^2 x = \frac{\cos^2 x}{\sin^2 x+\cos^2 x}\begin{matrix}:\cos^2 x\\ :\cos^2 x\end{matrix} = \frac{1}{\operatorname{tg}^2 x+1} = \frac{1}{1+t^2}$

$$I=\int \frac{dx}{3\cos^2 x+4\sin^2 x} = \left|\begin{matrix} \operatorname{tg} x = t & \sin^2 x = \frac{t^2}{1+t^2} \\ dx=\frac{dt}{1+t^2} & \cos^2 x = \frac{1}{1+t^2}\end{matrix}\right| =$$

$$=\int \frac{\frac{dt}{1+t^2}}{\frac{3}{1+t^2}+\frac{4t^2}{1+t^2}} = \int \frac{\frac{dt}{1+t^2}}{\frac{3+4t^2}{1+t^2}} = \int \frac{dt}{3+4t^2} = \int \frac{dt}{(\sqrt{3})^2+(2t)^2}$$

$$=\left|\begin{matrix} 2t=\sqrt{3}u \\ 2dt=\sqrt{3}du \\ dt=\frac{\sqrt{3}}{2}du \\ u=\frac{2t}{\sqrt{3}}\end{matrix}\right| = \int \frac{\frac{\sqrt{3}}{2}du}{3+3u^2} = \frac{\sqrt{3}}{2}\cdot\frac{1}{3}\int\frac{du}{1+u^2} = \frac{\sqrt{3}}{6}\operatorname{arc\,tg} u + C =$$

$$=\frac{\sqrt{3}}{6}\operatorname{arc\,tg}\frac{2t}{\sqrt{3}} + C = \frac{\sqrt{3}}{6}\operatorname{arc\,tg}\frac{2\operatorname{tg} x}{\sqrt{3}} + C$$

Odrediti $I=\int x^2 \sin x\, dx$.

Rj.

$$I=\int x^2\sin x\,dx = \left|\begin{matrix} u=x^2 & dv=\sin x\,dx \\ du=2x\,dx & v=-\cos x\end{matrix}\right| =$$

$$= -x^2\cos x - \int(-\cos x)\cdot 2x\,dx = -x^2 + 2\int x\cos x\,dx$$

$$\int x\cos x\,dx = \left|\begin{matrix} u=x & dv=\cos x\,dx \\ du=dx & v=\sin x\end{matrix}\right| = x\sin x - \int \sin x\,dx$$

$$= x\sin x - (-\cos x) + C = x\sin x + \cos x + C_1$$

$$I = -x^2\cos x + 2(x\sin x+\cos x + C_1) = -x^2\cos x + 2x\sin x + 2\cos x + C$$

$$= 2x\sin x - (x^2-2)\cos x + C$$

Izračunati integral $\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\,dx$.

Rj. Uvodimo smjenu $\operatorname{tg}\frac{x}{2}=t \Rightarrow \frac{x}{2} = \operatorname{arc\,tg} t$

$\sin 2x = 2\sin x\cos x$ $\qquad x = 2\operatorname{arc\,tg} t$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} =$ $\qquad dx = \frac{2}{1+t^2}$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\begin{matrix}/:\cos^2\frac{x}{2}\\ /:\cos^2\frac{x}{2}\end{matrix} = \frac{2\operatorname{tg}\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$\cos 2x = \cos^2 x - \sin^2 x$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\begin{matrix}/:\cos^2\frac{x}{2}\\ /:\cos^2\frac{x}{2}\end{matrix} =$$

$$= \frac{1-\operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1} = \frac{1-t^2}{1+t^2}$$

$$\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\,dx = \left| \begin{array}{ll} tg\frac{x}{2}=t & \sin x = \frac{2t}{1+t^2} \\ dx=\frac{2}{1+t^2} & \cos x=\frac{1-t^2}{1+t^2}\\ x=2\,arc\,tg\,t & \end{array}\right| =$$

$$=\int \frac{1-\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{\frac{1+\cancel{t^2}-2t+1-\cancel{t^2}}{\cancel{1+t^2}}}{\frac{\cancel{1}+t^2+2t-\cancel{1}+t^2}{\cancel{1+t^2}}}\cdot\frac{2}{1+t^2}\,dt$$

$$=2\int\frac{2-2t}{2t^2+2t}\cdot\frac{1}{1+t^2}\,dt = 2\int\frac{1-t}{(t^2+t)(1+t^2)}\,dt = 2\int\frac{1-t}{t(t+1)(t^2+1)}\,dt$$

$$\frac{1-t}{t(t+1)(t^2+1)} = \frac{A}{t}+\frac{B}{t+1}+\frac{Ct+D}{t^2+1} \quad /\cdot t(t+1)(t^2+1)$$

$$-t+1 = A\underbrace{(t+1)(t^2+1)}_{t^3+t^2+t+1} + B\underbrace{(t^2+1)\cdot t}_{t^3+t} + (Ct+D)\underbrace{t(t+1)}_{t^2+t}$$

$$-t+1 = A(t^3+t^2+t+1)+B(t^3+t)+C(t^3+t^2)+D(t^2+t)$$

$$\begin{array}{l} A+B+C=0\\ A+C+D=0\\ A+B+D=-1\\ A=1 \end{array} \qquad \begin{array}{ll} B+C=-1 & (a)\\ C+D=-1 & (b)\\ B+D=-2 & (c)\\ \hline -1+D=-2 & \\ D=-1 & \end{array} \qquad \begin{array}{ll} (a): & B+C=-1\\ (c)-(b): & B-C=-1 \quad +\\ \hline & 2B=-2\\ & B=-1 \end{array}$$

$$C-1=-1$$
$$C=0$$

$A=1$ $\quad C=0$

$B=-1$ $\quad D=-1$

$$2\int\frac{1-t}{t(t+1)(t^2+1)}\,dt = 2\int\left(\frac{1}{t}+\frac{(-1)}{t+1}+\frac{(-1)}{t^2+1}\right)dt =$$

$$=2\ln|t|-2\ln|t+1|-2\,arc\,tg\,t+C=$$

$$=2\ln\left|tg\frac{x}{2}\right|-2\ln\left|tg\frac{x}{2}+1\right|-2\,arc\,tg\left|tg\frac{x}{2}\right|+C$$

Dio tablice integrala

1. $\int u^a\,du=\frac{u^{a+1}}{a+1}+C,\ a\neq-1.$

2. $\int u^{-1}\,du=\int\frac{du}{u}=\int\frac{u'}{u}\,dx=$
$=\ln|u|+C.$

3. $\int a^u\,du=\frac{a^u}{\ln a}+C;\ \int e^u\,du=$
$=e^u+C.$

4. $\int \sin u\,du=-\cos u+C.$

5. $\int \cos u\,du=\sin u+C.$

6. $\int \sec^2 u\,du=\mathrm{tg}\,u+C.$

7. $\int \mathrm{cosec}^2 u\,du=-\mathrm{ctg}\,u+C.$

8. $\int\frac{du}{u^2+a^2}=\frac{1}{a}\,\mathrm{arc\,tg}\,\frac{u}{a}+C.$

9. $\int\frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$

10. $\int\frac{du}{\sqrt{a^2-u^2}}=\mathrm{arc\,sin}\frac{u}{a}+C.$

11. $\int\frac{du}{\sqrt{u^2+a}}=$
$=\ln\left|u+\sqrt{u^2+a}\right|+C.$

Određeni integrali

Osobine određenih integrala su:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. $\int_a^b [f_1(x) + f_2(x) - f_3(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx - \int_a^b f_3(x)\,dx$

5. $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$, gdje je α konstanta

Određene integrale ćemo računati pomoću Njutn-Lejbnicove formule

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big|_a^b = F(x)\Big|_a^b = F(b) - F(a)$$

gdje je $F'(x) = f(x)$

(#) Izračunati integrale

a) $\int_2^3 3x^2\,dx$; b) $\int_0^4 (1+e^{\frac{x}{4}})\,dx$; c) $\int_{-1}^7 \frac{dt}{\sqrt{3t+4}}$;

d) $\int_0^{\pi/2a} (x+3)\sin ax$

Rj. a) $$\int_2^3 3x^2\,dx = 3\int_2^3 x^2\,dx = 3\cdot\frac{x^3}{3}\Big|_2^3 = x^3\Big|_2^3 = 3^3 - 2^3 = 27-8 = 19$$

b) $$\int_0^4 (1+e^{\frac{x}{4}})\,dx = \int_0^4 dx + \int_0^4 e^{\frac{x}{4}}\,dx = \int_0^4 dx + 4\int_0^4 e^{\frac{x}{4}}\,d\left(\frac{x}{4}\right) =$$

$$= x\Big|_0^4 + 4\,e^{\frac{x}{4}}\Big|_0^4 = (4-0) + 4\left(e^{\frac{4}{4}} - e^{\frac{0}{4}}\right) = 4 + 4e - 4 = 4e$$

c) $$\int_{-1}^7 \frac{dt}{\sqrt{3t+4}} = \int_{-1}^7 (3t+4)^{-\frac{1}{2}}\,dt = \left| \begin{array}{l} d(3t+4) = 3\,dt \\ dt = \frac{1}{3}\,d(3t+4) \end{array} \right| =$$

$$= \frac{1}{3}\int_{-1}^7 (3t+4)^{-\frac{1}{2}}\,d(3t+4) = \frac{2}{3}(3t+4)^{\frac{1}{2}}\Big|_{-1}^7 = \frac{2}{3}\left(\sqrt{25} - \sqrt{1}\right) = \frac{8}{3}$$

d) $$\int_0^{\frac{\pi}{2a}} (x+3)\sin ax\,dx = \left| \begin{array}{ll} u = x+3 & dv = \sin ax\,dx \\ du = dx & v = \frac{1}{a}\int \sin ax\,d(ax) = \frac{-1}{a}\cos ax \end{array} \right| =$$

$$= -\frac{1}{a}(x+3)\cos ax\Big|_0^{\frac{\pi}{2a}} + \frac{1}{a}\int_0^{\pi/2a} \cos ax\,dx = -\frac{1}{a}\Big[\left(\tfrac{\pi}{2a}+3\right)\underbrace{\cos\tfrac{\pi}{2}}_{=0} - 3\underbrace{\cos 0}_{=1}\Big] +$$

$$+ \frac{1}{a}\cdot\frac{1}{a}\int_0^{\pi/2a} \cos ax\,d(ax) = \frac{3}{a} + \frac{1}{a^2}\underbrace{\sin ax\Big|_0^{\frac{\pi}{2a}}}_{\sin\frac{\pi}{2} - \sin 0} = \frac{3}{a} + \frac{1}{a^2} = \frac{1+3a}{a^2}$$

Zadaci za vežbu

1. $\displaystyle\int_1^5 \frac{dx}{3x-2}$

2. $\displaystyle\int_0^1 \frac{dz}{(2z+1)^3}$

3. $\displaystyle\int_1^2 \frac{dt}{t^2+5t+4}$

4. $\displaystyle\int_0^2 \frac{x+3}{x^2+4}\,dx$

5. $\displaystyle\int_{-a}^{a} x\cos\frac{x}{a}\,dx$

6. $\displaystyle\int_0^{\pi} \cos\frac{x}{2}\cos\frac{3x}{2}\,dx$

7.* $\displaystyle\int_{-\pi}^{\pi} x\sin x\cos x\,dx$

8. $\displaystyle\int_1^{e} (1+\ln y)^2\,dy$

Rešenja:

1. $\frac{\ln 13}{3}$ 2. $\frac{2}{9}$ 3. $\frac{1}{3}\ln\frac{5}{4}$ 4. $\frac{3\pi}{8}+\frac{\ln 2}{2}$

5. 0 6. 0 7. $-\frac{\pi}{2}$ 8. $2e-1$

Zamena promenljivih u određenom integralu

$$\int_a^b f(x)\,dx = \left| \begin{matrix} x=\varphi(t) & x=a \Rightarrow a=\varphi(\alpha) \Rightarrow & t=\alpha \\ dx=\varphi'(t)\,dt & x=b \Rightarrow b=\varphi(\beta) \Rightarrow & t=\beta \end{matrix} \right|$$

$$= \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt = \int_\alpha^\beta F(t)\,dt$$

Izračunati integrale

a) $\displaystyle\int_0^5 \frac{x\,dx}{\sqrt{1+3x}}$; b) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{dx}{e^x-e^{-x}}$; c) $\displaystyle\int_1^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}}$; d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

Rj.

a) $$\int_0^5 \frac{x\,dx}{\sqrt{1+3x}} = \left| \begin{matrix} 1+3x=t^2 & 3\,dx=2t\,dt \\ \sqrt{1+3x}=t & dx=\frac{2}{3}t\,dt \\ 3x=t^2-1 & \\ x=\frac{t^2-1}{3} & x\Big|_0^5 \Rightarrow t\Big|_1^4 \end{matrix} \right| = \int_1^4 \frac{\frac{t^2-1}{3}\cdot\frac{2}{3}t\,dt}{t} =$$

$$= \frac{2}{9}\int_1^4 (t^2-1)\,dt = \frac{2}{9}\left(\frac{t^3}{3}\Big|_1^4 - t\Big|_1^4\right) = \frac{2}{9}\left(\frac{1}{3}(64-1)-(4-1)\right) =$$

$$= \frac{2}{9}\left(\frac{63}{3}-3\right) = \frac{2}{9}(21-3) = 4$$

b) $$\int_{\ln 2}^{\ln 3} \frac{dx}{e^x-e^{-x}} = \left| \begin{matrix} e^x=t & e^{-x}=t^{-1}=\frac{1}{t} \\ x=\ln t & \\ dx=\frac{dt}{t} & x\Big|_{\ln 2}^{\ln 3} \Rightarrow t\Big|_2^3 \end{matrix} \right| = \int_2^3 \frac{\frac{dt}{t}}{t-\frac{1}{t}} = \int_2^3 \frac{\frac{dt}{t}}{\frac{t^2-1}{t}} =$$

$$= \int_2^3 \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\Bigg|_2^3 = \frac{1}{2}\left(\ln\frac{2}{4}-\ln\frac{1}{3}\right) = \frac{1}{2}\cdot\ln\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\ln\frac{3}{2}}{2}$$

c) $$\int_1^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}} = \left| \begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \quad x\Big|_1^{\sqrt{3}} \Rightarrow t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ x^3 = 8\sin^3 t \\ \sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = \sqrt{4(1-\sin^2 t)} \end{array} \right| =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(8\sin^3 t+1)\,2\cos t\,dt}{4\sin^2 t\sqrt{4\cos^2 t}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin^3 t+1}{4\sin^2 t}\,dt =$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin t\,dt + \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dt}{\sin^2 t} = -2\cos t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \frac{1}{4}\,\mathrm{ctg}\,t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$= -2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) - \frac{1}{4}\left(\frac{\sqrt{3}}{3} - \sqrt{3}\right) = \frac{7}{2\sqrt{3}} - 1$$

d) $$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \left| \begin{array}{ll} \mathrm{tg}\,\frac{x}{2} = z & dx = \frac{2dz}{1+z^2} \\ \cos x = \frac{1-z^2}{1+z^2} & x\Big|_0^{\frac{\pi}{2}} \Rightarrow z\Big|_0^1 \end{array} \right| =$$

$$= \int_0^1 \frac{\frac{2dz}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} = 2\int_0^1 \frac{dz}{z^2+3} = \frac{2}{\sqrt{3}}\,\mathrm{arc\,tg}\,\frac{z}{\sqrt{3}}\Big|_0^1 =$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

(#) Dokazati da za parnu f-ju $f(x)$ vrijedi

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

dok za neparnu f-ju $f(x)$ vrijedi $\int_{-a}^{a} f(x)\,dx = 0$.

Rj. Prvo rastavimo interval $[-a,a]$ na dva dijela $[-a,0]$ i $[0,a]$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \quad \dots(*)$$

Posmatrajmo sad $\int_{-a}^{0} f(x)\,dx$. Ako uvedemo smjenu $x=-z$ imamo da je $dx=-dz$ i $z_1=a$ za $x_1=-a$, $z_2=0$ za $x_2=0$

$$\int_{-a}^{0} f(x)\,dx = -\int_a^0 f(-z)\,dz = \int_0^a f(-z)\,dz = \left|\begin{array}{c}\text{nove promj.} \\ x=z\end{array}\right| = \int_0^a f(-x)\,dx$$

Prema tome $(*)$ sad postaje

$$I = \int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx = \int_0^a [f(x)+f(-x)]\,dx$$

Za parnu f-ju znamo da $f(-x)=f(x)$ dok je za neparnu f-ju $f(-x)=-f(x)$. Prema tome

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{ako je } f(x) \text{ parna f-ja} \\ 0, & \text{ako je } f(x) \text{ neparna f-ja} \end{cases}$$

Znamo da za parnu f-ju $f(x)$ vrijedi

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx,$$

dok za neparnu f-ju $f(x)$ vrijedi: $\int_{-a}^{a} f(x)\,dx = 0$.
Iskoristiti ovu osobinu i izračunati sljedeće integrale:

a) $\int_{-\sqrt{5}}^{\sqrt{5}} (3x-2x^5)\,dx$ b) $\int_{-\pi}^{\pi} \sin^7 2x\,dx$ c) $\int_{3}^{-3} t^8 \arcsin t\,dt$

d) $\int_{-2}^{2} \frac{x^5+7x^4+x^3-5x^2-2}{x^3+x}\,dx$

Rj:

a) $f(x) = 3x-2x^5$

$f(-x) = 3(-x)-2(-x)^5 = -3x+2x^5 = -(3x-2x^5) = -f(x)$

$$\int_{-\sqrt{5}}^{\sqrt{5}} (3x-2x^5)\,dx = \left| \begin{array}{l} \text{primijetimo} \\ \text{da je} \\ 3x-2x^5 \text{ neparna} \\ \text{f-ja} \end{array} \right| = 0$$

b) $f(x) = \sin^7 2x \Rightarrow f(-x) = (\sin 2(-x))^7 = (-\sin 2x)^7 = -\sin^7 2x = -f(x)$

Kako je $\sin^7 2x$ neparna f-ja $\int_{-\pi}^{\pi} \sin^7 2x\,dx = 0$

c) $\int_{3}^{-3} t^8 \arcsin t\,dt = 0$ ZAŠTO? OBJASNITI!

d) $$\int_{-2}^{2} \frac{x^5+7x^4+x^3-5x^2-2}{x^3+x}\,dx = \int_{-2}^{2} \frac{\overbrace{x^5+x^3}^{x^2(x^3+x)}}{x^3+x}\,dx + \int_{-2}^{2} \frac{7x^4-5x^2-2}{x^3-x}\,dx =$$

$$= \int_{-2}^{2} x^2\,dx + 0 = 2\int_{0}^{2} x^2\,dx = 2\,\frac{x^3}{3}\Big|_0^2 = \frac{16}{3}$$

Zadaci za vježbu

Izračunati integrale

1. $\int_0^1 \frac{x^2\,dx}{(x+1)^4}$ uvođenjem smjene $x+1=z$.

2. $\int_0^{\ln 2} \sqrt{e^x-1}\,dx$ uvođenjem smjene $\sqrt{e^x-1}=t$.

3. $\int_{\sqrt{3}}^{\sqrt{7}} \frac{x^3\,dx}{\sqrt[3]{(x^2+1)^2}}$ uvođenjem smjene $z=x^2+1$.

4. $\int_1^e \frac{\sqrt[4]{1+\ln x}}{x}\,dx$ uvođenjem smjene $t=1+\ln x$.

5. $\int_{-3}^{3} x^2\sqrt{9-x^2}\,dx$ uvođenjem smjene $x=3\cos\varphi$

6. $\int_5^1 \frac{t\,dt}{\sqrt{5+4t}}$ 7. $\int_0^{\frac{\pi}{4}} \frac{1+tg^2\varphi}{1+tg\,\varphi}\,d\varphi$ 8. $\int_{\ln 3}^{0} \frac{1-e^x}{1+e^x}\,dx$

9. $\int_{-1}^{0} \frac{dx}{1+\sqrt[3]{x+1}}$ 10.* $\int_0^8 \sqrt{\frac{x}{6-x}}\,dx$ 11.* $\int_0^{\frac{\pi}{2}} \sin^3\varphi\sqrt{\cos\varphi}\,d\varphi$

Rješenja:

1. $\frac{1}{24}$ 2. $\frac{4-\pi}{2}$ 3. 3 4. $0{,}8(2\sqrt[4]{2}-1)$ 5. $\frac{81\pi}{8}$ 6. $-\frac{17}{6}$

7. $\ln 2$ 8. $\ln\frac{4}{3}$ 9. $\frac{3}{2}(\ln 4-1)$ 10. $\frac{3(\pi-2)}{2}$ (uvodimo smjenu $x=6\sin^2 t$)

11. $8/21$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Računje određenih integrali i Smjena promjenjivih u određenim integralima)

1. $$\int_{\pi/6}^{\pi/3} \frac{dx}{\cos^2 x} = \operatorname{tg} x \Big|_{\pi/6}^{\pi/3} = \operatorname{tg}\frac{\pi}{3} - \operatorname{tg}\frac{\pi}{6} = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

2. $$\int_{\pi/4}^{\pi/3} \sin x\, dx = -\cos x \Big|_{\pi/4}^{\pi/3} = -\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{4}\right) = -\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) = -\frac{1-\sqrt{2}}{2}$$

3. $$\int_{1/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{1/2}^{\sqrt{3}/2} = \arcsin\frac{\sqrt{3}}{2} - \arcsin\frac{1}{2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

4. $$\int_a^b x^m\, dx = \frac{x^{m+1}}{m+1}\Big|_a^b = \frac{1}{m+1}\left(b^{m+1} - a^{m+1}\right)$$

5. $$\int_0^1 (e^x-1)^4 e^x\, dx = \left| \begin{array}{l} e^x - 1 = t \\ e^x dx = dt \\ x=0 \Rightarrow t=0 \\ x=1 \Rightarrow t=e-1 \end{array} \right| = \int_0^{e-1} t^4\, dt = \frac{t^5}{5}\Big|_0^{e-1} = \frac{1}{5}(e-1)^5$$

6. $$\int_2^9 \sqrt[3]{x-1}\, dx = \left| \begin{array}{l} x-1 = t^3 \\ dx = 3t^2 dt \\ x=2 \Rightarrow t=1 \\ x=9 \Rightarrow t=2 \end{array} \right| = \int_1^2 \sqrt[3]{t^3}\cdot 3t^2\, dt = 3\int_1^2 t^3\, dt = \frac{3}{4}t^4\Big|_1^2 =$$
$$= \frac{3}{4}(16-1) = \frac{45}{4}$$

7. $$\int_0^{\ln 5} \frac{\sqrt{e^x-1}}{1+3e^{-x}}\, dx \, \frac{\cdot e^x}{\cdot e^x} = \int_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\, dx = \left| \begin{array}{l} e^x - 1 = t^2 \\ e^x dx = 2t\,dt \\ x=0 \Rightarrow t=0 \\ x=\ln 5 \Rightarrow t=2 \\ e^x = t^2+1 \end{array} \right| = \int_0^2 \frac{\sqrt{t^2}\cdot 2t}{t^2+1+3}\, dt$$
$$= 2\int_0^2 \frac{t^2+4-4}{t^2+4}\, dt = 2\int_0^2 dt - 2\int_0^2 \frac{4}{t^2+4}\, dt = 2t\Big|_0^2 - 8\cdot\frac{1}{2}\operatorname{arctg}\frac{t}{2}\Big|_0^2 =$$
$$= 4 - 4(\operatorname{arctg} 1 - \operatorname{arctg} 0) = 4 - 4\cdot\frac{\pi}{4} = 4 - \pi$$

Osobine određenih integrala

a) $\int_a^a f(x)\, dx = 0$

b) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

c) $\int_a^b c\cdot f(x)\, dx = c\int_a^b f(x)\, dx$

d) $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

e) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \quad \forall x$

8. $$\int_0^{\sqrt{7}} \frac{dx}{7+x^2} = \frac{1}{\sqrt{7}}\operatorname{arctg}\frac{x}{\sqrt{7}} = \frac{1}{\sqrt{7}}\left(\operatorname{arctg}\frac{\sqrt{7}}{\sqrt{7}} - \operatorname{arctg}\frac{0}{\sqrt{7}}\right) = \frac{1}{\sqrt{7}}\cdot\frac{\pi}{4} = \frac{\sqrt{7}\pi}{28}$$

9. $$\int_0^{1/2} \sqrt{1-x^2}\, dx = \left| \begin{array}{l} x = \sin t \\ dx = \cos t\, dt \\ x=0 \Rightarrow \sin t = 0 \Rightarrow t=0 \\ x=\frac{1}{2} \Rightarrow \sin t = \frac{1}{2} \Rightarrow t=\frac{\pi}{6} \end{array} \right| = \int_0^{\pi/6} \sqrt{1-\sin^2 t}\,\cos t\, dt =$$

kako je $\sin^2 t + \cos^2 t = 1$

$$= \int_0^{\pi/6} \sqrt{\cos^2 t}\,\cos t\, dt = \int_0^{\pi/6} \cos^2 t\, dt = \frac{1}{2}\int_0^{\pi/6} (1+\cos 2t)\, dt = \frac{1}{2}t\Big|_0^{\pi/6} + \frac{1}{4}\sin 2t\Big|_0^{\pi/6}$$

$1 = \cos^2 t + \sin^2 t$
$\cos 2t = \cos^2 t - \sin^2 t$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} = \frac{\pi}{12} + \frac{\sqrt{3}}{8} = \frac{2\pi + 3\sqrt{3}}{24}$$

10. $$\int_0^4 \frac{dx}{1+\sqrt{2x+1}}$$
uputa: smjena $2x+1 = t^2$
rj. $2 - \ln 2$

11. $$\int_0^1 \frac{dx}{\sqrt{2-x^2+x}}$$
uputa: $-x^2+x+2 = \ldots = \frac{9}{4} - \left(x-\frac{1}{2}\right)^2$
$x - \frac{1}{2} = \frac{3}{2}t$
rj. $2\arcsin\frac{1}{3}$

12. $$\int_1^e x\ln x\, dx = \left| \begin{array}{ll} u = \ln x & dv = x\,dx \\ du = \frac{dx}{x} & v = \frac{x^2}{2} \end{array} \right| = \frac{1}{2}x^2\ln x\Big|_1^e - \frac{1}{2}\int_1^e x^2\cdot\frac{dx}{x} =$$
$$= \frac{1}{2}\left(e^2\ln e - 1^2\ln 1\right) - \frac{1}{2}\int_1^e x\, dx = \frac{1}{2}e^2 - \frac{1}{2}\cdot\frac{1}{2}x^2\Big|_1^e =$$
$$= \frac{1}{2}e^2 - \frac{1}{4}(e^2-1) = \frac{1}{4}e^2 + \frac{1}{4}$$

Izračunati integral $\displaystyle\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5}\,dx$.

Rj. $x-4\sqrt{x}+5 = x-2\cdot\sqrt{x}\cdot 2+4+1=(\sqrt{x}-2)^2+1$

$$\int_1^4 \frac{\sqrt{x}+2}{(\sqrt{x}-2)^2+1}\,dx = \left| \begin{array}{l} x=t^2 \\ dx=2t\,dt \\ x=1 \Rightarrow t=1 \\ x=4\Rightarrow t=2\end{array}\right| = \int_1^2 \frac{t+2}{(t-2)^2+1}\cdot 2t\,dt =$$

$$=2\int_1^2 \frac{t^2+2t}{t^2-4t+5}\,dt = 2\int_1^2 \frac{t^2+2t-6t+6t+5-5}{t^2-4t+5}\,dt = 2\int_1^2\frac{t^2-4t+5}{t^2-4t+5}\,dt+$$

$$+2\int_1^2\frac{6t-5}{t^2-4t+5}\,dt = 2\int_1^2 dt + 2\cdot 3\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}$$

$$\int_1^2\frac{2t-\frac53}{t^2-4t+5}\,dt = \int_1^2 \frac{2t-4+4-\frac53}{t^2-4t+5}\,dt = \int_1^2\frac{2t-4}{t^2-4t+5}\,dt+\frac73\int_1^2\frac{dt}{t^2-4t+5}$$

$$\int_1^2 dt = t\Big|_1^2 = 2-1=1\ , \quad \int_1^2 \frac{2t-4}{t^2-4t+5}\,dt = \left|\begin{array}{l} t^2-4t+5=s \\ (2t-4)\,dt=ds \\ t=1 \Rightarrow s=2 \\ t=2 \Rightarrow s=1\end{array}\right| = \int_2^1 \frac{ds}{s} = \ln|s|\Big|_2^1 = \ln 1-\ln 2=-\ln 2$$

$$\int_1^2\frac{dt}{t^2-4t+5} = \int_1^2\frac{dt}{(t-2)^2+1} = \left|\begin{array}{l} t-2=s \\ dt=ds \\ t=1 \Rightarrow s=-1 \\ t=2 \Rightarrow s=0\end{array}\right| = \int_{-1}^0\frac{ds}{s^2+1} = \operatorname{arctg} s\Big|_{-1}^0 = 0-\left(-\frac{\pi}{4}\right)=\frac{\pi}{4}$$

$$\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5} = 2\cdot 1+6\left(-\ln 2+\frac73\cdot\frac\pi4\right) = 2-6\ln 2+\frac{7\pi}{2}\approx 8{,}8367$$

traženo rješenje

Izračunati integral $\displaystyle\int_0^{\pi/4}\sin^5x\,\cos^7x\,dx$

Rj. $$\int_0^{\frac\pi4}\sin^5x\cdot\cos^7x\,dx = \int_0^{\frac\pi4}\sin^5x\cdot\cos^6x\cdot\cos x\,dx = \left|\begin{array}{l} \sin x=t \\ \cos x\,dx=dt \\ x=0 \Rightarrow t=0 \\ x=\frac\pi4 \Rightarrow t=\frac{\sqrt2}{2} \\ \cos^6x=(\cos^2x)^3= \\ =(1-\sin^2x)^3=(1-t^2)^3\end{array}\right| =$$

$$=\int_0^{\frac{\sqrt2}{2}} t^5(1-t^2)^3\,dt =$$

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \end{array}$$

$$=\int_0^{\frac{\sqrt2}{2}} t^5(1-3t^2+3t^4-t^6)\,dt = \int_0^{\frac{\sqrt2}{2}}(t^5-3t^7+3t^9-t^{11})\,dt =$$

$$=\frac16 t^6\Big|_0^{\frac{\sqrt2}{2}} - \frac38 t^8\Big|_0^{\frac{\sqrt2}{2}}+\frac3{10}t^{10}\Big|_0^{\frac{\sqrt2}{2}}-\frac1{12}t^{12}\Big|_0^{\frac{\sqrt2}{2}}=$$

$$=\frac{1}{\cancel{6}_3}\cdot\frac{\cancel{8}^{\cancel{4}\,1}}{\cancel{64}_{16}} - \frac{3}{\cancel{8}_1}\cdot\frac{\cancel{16}^{\cancel{2}\,1}}{\cancel{256}_{128}}+\frac{3}{\cancel{10}_5}\cdot\frac{\cancel{32}^{\cancel{16}\,1}}{\cancel{1024}_{64}}-\frac{1}{\cancel{12}_3}\cdot\frac{\cancel{64}^{\cancel{16}\,1}}{\cancel{4096}_{256}}=$$

$$=\frac{1}{\underbrace{3\cdot16}_{3\cdot2^4}}-\frac{3}{\underbrace{128}_{2^7}}+\frac{3}{\underbrace{5\cdot64}_{5\cdot2^6}}-\frac{1}{\underbrace{3\cdot256}_{3\cdot2^8}} = \frac{5\cdot2^4-3\cdot3\cdot5\cdot2+3\cdot3\cdot2^2-5}{3\cdot5\cdot2^8}$$

$$=\frac{80-90+36-5}{3\cdot5\cdot2^8} = \frac{2\cancel{1}^{7}}{\cancel{3}_1\cdot5\cdot2^8}=\frac{7}{1280}$$

traženo rješenje

Izračunati integral $\displaystyle\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx$

Rj.

Metoda Ostrogradskog:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = (ax^2+bx+c)\sqrt{x^2+3} + \lambda \int \frac{dx}{\sqrt{x^2+3}} \quad \Big/ \frac{d}{dx}$$

$$\frac{2x^3-7x+4}{\sqrt{x^2+3}} = (2ax+b)\sqrt{x^2+3} + (ax^2+bx+c)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+3}} + \frac{\lambda}{\sqrt{x^2+3}} \quad \Big/ \cdot \sqrt{x^2+3}$$

$$2x^3-7x+4 = (2ax+b)(x^2+3) + (ax^2+bx+c)x + \lambda$$

$$2x^3-7x+4 = 2ax^3+bx^2+6ax+3b+ax^3+bx^2+cx+\lambda$$

x^3: $2a+a=2 \Rightarrow 3a=2$, $a=\frac{2}{3}$

x^2: $b+b=0 \Rightarrow b=0$

x: $6a+c=-7 \Rightarrow 6\cdot\frac{2}{3}+c=-7$, $4+c=-7$, $c=-11$

x^0: $3b+\lambda=4$, $\lambda=4$

Prema tome:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \left(\frac{2}{3}x^2-11\right)\sqrt{x^2+3} + 4\int\frac{dx}{\sqrt{x^2+3}} =$$

$$= \frac{2}{3}x^2\sqrt{x^2+3} - 11\sqrt{x^2+3} + 4\ln\left|x+\sqrt{x^2+3}\right| + C$$

Prema tome

$$\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \frac{2}{3}x^2\sqrt{x^2+3}\Big|_{-1}^{1} - 11\sqrt{x^2+3}\Big|_{-1}^{1} + 4\ln\left|x+\sqrt{x^2+3}\right|\Big|_{-1}^{1} =$$

$$= \frac{2}{3}(2-2) - 11(2-2) + 4\left(\ln|1+2| - \ln|-1+2|\right) =$$

$$= 4(\ln 3 - \ln 1) = 4\ln 3$$ traženi rezultat

Izračunati: $\displaystyle\int_{3}^{4} \frac{6x+8}{x^2+x-6}\,dx$

Rj.

$$\frac{6x+8}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3} \quad \Big/ (x-2)(x+3)$$

$D = 1+24 = 25$

$x_{1,2} = \frac{-1\pm 5}{2}$

$x_1=-3 \quad x_2=2$

$x^2+x-6 = (x-2)(x+3)$

$6x+8 = A(x+3)+B(x-2)$

$6x+8 = (A+B)x + (3A-2B)$

$A+B=6 \quad /\cdot 2$

$3A-2B=8$

$2A+2B=12$

$+\ 3A-2B=8$

$5A=20$

$A=4$

$A+B=6$

$4+B=6$

$B=2$

$$\int\frac{6x+8}{x^2+x-6}\,dx = \int\left(\frac{4}{x-2}+\frac{2}{x+3}\right)dx = 4\int\frac{dx}{x-2} + 2\int\frac{dx}{x+3} =$$

$$= 4\ln|x-2| + 2\ln|x+3| + C$$

$$\int_3^4 \frac{6x+8}{x^2+x-6}\,dx = 4\ln|x-2|\Big|_3^4 + 2\ln|x+3|\Big|_3^4 = 4(\ln 2 - \underbrace{\ln 1}_{=0}) +$$

$$+ 2(\ln 7 - \ln 6) = 4\ln 2 + 2\ln\frac{7}{6} = \ln 2^4 + \ln\left(\frac{7}{6}\right)^2$$

$$= \ln\frac{7^2}{2^2\cdot 3^2}\cdot 2^4 = \ln\frac{49\cdot 4}{9} = \ln\frac{196}{9}$$

$$\int_3^4 \frac{6x+8}{x^2+x-6}\,dx = \ln\frac{196}{9}$$ traženo rješenje

(#) Izračunati integral $I=\int_0^1 \arcsin\frac{x}{2}\,dx$.

Rj.
$$\int \arcsin\frac{x}{2}\,dx = \left|\begin{array}{l} \frac{x}{2}=t \\ \frac{1}{2}dx=dt \\ dx=2dt \end{array}\right| = 2\int \arcsin t\,dt = \left|\begin{array}{ll} u=\arcsin t & dv=dt \\ du=\frac{dt}{\sqrt{1-t^2}} & v=t \end{array}\right| =$$

$$= 2\left(t\arcsin t - \int \frac{t}{\sqrt{1-t^2}}dt\right) = 2t\arcsin t - \int \frac{2t\,dt}{\sqrt{1-t^2}} \underset{=}{\overset{(**)}{=}}$$

$$\int \frac{-2t\,dt}{\sqrt{1-t^2}} = \left|\begin{array}{l} 1-t^2=s \\ -2t\,dt=ds \end{array}\right| = \int \frac{ds}{\sqrt{s}} = \int s^{-\frac{1}{2}}ds = \frac{s^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{s}+C =$$
$$= 2\sqrt{1-t^2}+C$$

$$\overset{(**)}{=} 2t\arcsin t + 2\sqrt{1-t^2}+C = x\arcsin\frac{x}{2} + 2\sqrt{1-\frac{x^2}{4}} + C$$

$$\int_0^1 \arcsin\frac{x}{2}dx = x\arcsin\frac{x}{2}\Big|_0^1 + 2\sqrt{1-\frac{x^2}{4}}\Big|_0^1 = \arcsin\frac{1}{2} + \left(2\sqrt{1-\frac{1}{4}} - 2\right) =$$

$$= \frac{\pi}{6} + \frac{2\sqrt{3}}{2} - 2 = \frac{\pi}{6} + \sqrt{3} - 2$$

Nepravi integrali

Nepravi integral ugraničena od a do $+\infty$ je oblika
$$I=\int_a^{+\infty} f(x)\,dx$$
Rješavamo ga na sljedeći način:
$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx$$

Ako postoji
$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$
kažemo da integral konvergira ili da postoji nepravi integral, a ako limes ne postoji (kao realan broj), kažemo da integral divergira ili da nepravi integral ne postoji.

(1.) Izračunati:

a) $$\int_1^{+\infty}\frac{dx}{x^4} = \lim_{a\to+\infty}\int_1^a \frac{dx}{x^4} = \lim_{a\to+\infty}\int_1^a x^{-4}dx = \lim_{a\to+\infty}\frac{x^{-3}}{-3}\Big|_1^a =$$
$$= -\frac{1}{3}\lim_{a\to\infty}\frac{1}{x^3}\Big|_1^a = -\frac{1}{3}\lim_{a\to\infty}\left(\overset{\nearrow 0}{\frac{1}{a^3}} - 1\right) = \left(-\frac{1}{3}\right)(-1) = \frac{1}{3}$$

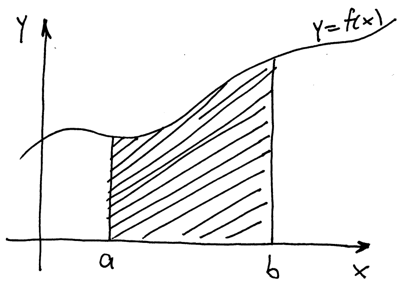
b) $$\int_1^{+\infty}\frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b\frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b x^{-\frac{1}{2}}dx = \lim_{b\to\infty}\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Big|_1^b = 2\lim \sqrt{x}\Big|_1^b$$
$$= 2\lim_{b\to\infty}(\sqrt{b}-1) = +\infty,$$ integral divergira

c) $$\int_0^{+\infty} e^{-ax}dx = \lim_{t\to\infty}\int_0^t e^{-ax}dx = \left|\begin{array}{ll} -ax=s & x=0 \Rightarrow s=0 \\ -a\,dx=ds & x=t \Rightarrow s=-at \\ dx=-\frac{1}{a}ds & \end{array}\right| =$$
$(a>0)$
$$= \lim_{t\to\infty}\int_0^{-at} e^s\left(-\frac{1}{a}\right)ds = -\frac{1}{a}\lim_{t\to\infty} e^s\Big|_0^{-at} = -\frac{1}{a}\lim_{t\to\infty}\left(\overset{\nearrow 0}{e^{-at}} - 1\right) = \frac{1}{a}$$

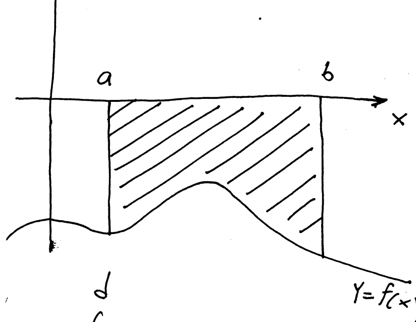
d) $\int_2^{+\infty}\frac{\ln x}{x}dx$ Rj. divergira. e) $\int_1^{+\infty}\frac{dx}{x^2(x+1)}$ Rj. $1-\ln 2$ f) $\int_0^{+\infty} xe^{-x^2}dx$ Rj. $\frac{1}{2}$

Primjena određenog integrala

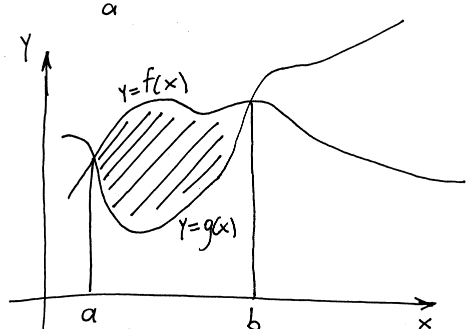
I Izračunavanje površine ravne figure



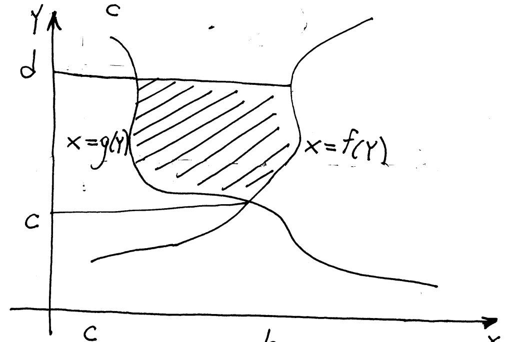
$$P=\int_a^b f(x)\,dx$$



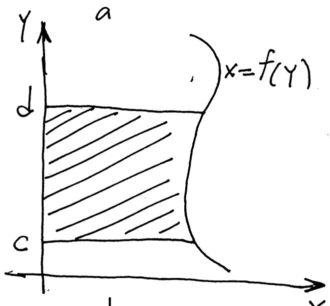
$$P=-\int_a^b f(x)\,dx$$



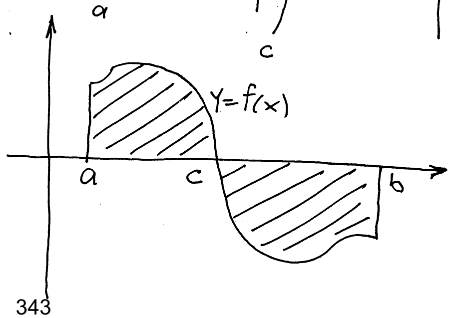
$$P=\int_a^b [f(x)-g(x)]\,dx$$



$$P=\int_c^d [f(y)-g(y)]\,dy$$



$$P=\int_c^d f(y)\,dy$$



$$P=\int_a^c f(x)\,dx+\left|\int_c^b f(x)\,dx\right|$$

1. Izračunati površinu ravne figure koja je ograničena linijama $y=4-(x-2)^2$ i $y=0$.

Rj.

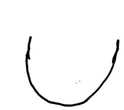
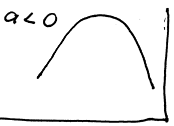
$y=4-(x-2)^2$

$y=4-(x^2-4x+4)$

$y=-x^2+4x$

$y=-x(x-4)$

Kriva $y=ax^2+bx+c$ ima grafik u obliku parabole. Tjeme parabole $T\left(-\frac{b}{2a},\frac{4ac-b^2}{4a}\right)$

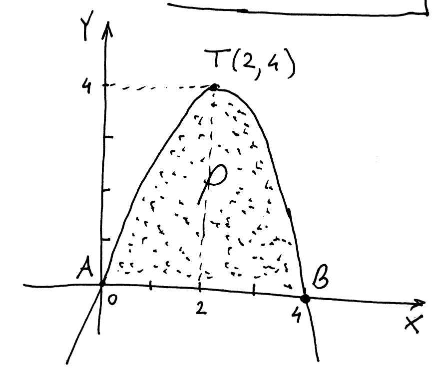
za $a>0$  za $a<0$ 

Nule $A(0,0)$ i $B(4,0)$

$$-\frac{b}{2a}=-\frac{4}{2\cdot(-1)}=2$$

$$\frac{4ac-b^2}{4a}=\frac{0-16}{-4}=4$$

Tjeme parabole $y=4-(x-2)^2$ je u tački $(2,4)$.



$$P=\int_0^4(-x^2+4x)\,dx=\int_0^4(-x^2)\,dx+\int_0^4 4x\,dx=-\frac{x^3}{3}\Big|_0^4+4\cdot\frac{x^2}{2}\Big|_0^4=-\frac{1}{3}(4^3-0^3)$$

$$+2(4^2-0^2)=-\frac{1}{3}\cdot 64+32=\frac{96}{3}-\frac{64}{3}=\frac{32}{3}$$

2. Izračunati površinu ravne figure koja je ograničena krivom $y=x^2-4x+3$ i pravama $y=0$, $x=0$ i $x=2$.

Rj. $y=x^2-4x+3$

$D=16-12=4$

$x_{1,2}=\frac{4\pm 2}{2}$

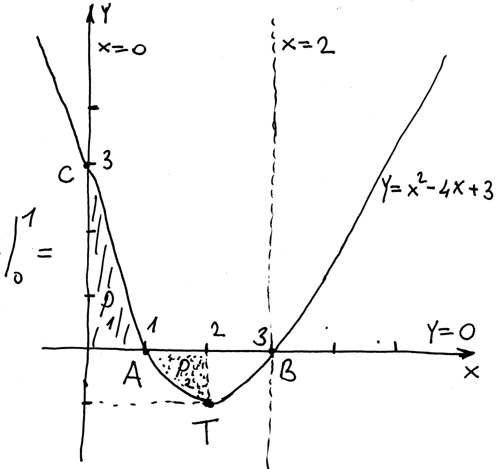
Nule krive $A(1,0)$ i $B(3,0)$

$$-\frac{b}{2a}=-\frac{-4}{2}=2$$

$$\frac{4ac-b^2}{4a}=\frac{12-16}{4}=-1$$

Tjeme krive $y=x^2-4x+3$ je u tački $T(2,-1)$.

C(0,3) je tačka presjeka krive sa Y-osom



$$P = P_1 + P_2$$

$$P_1 = \int_0^1 (x^2-4x+3)\,dx = \frac{x^3}{3}\Big|_0^1 - 4\cdot\frac{x^2}{2}\Big|_0^1 + 3x\Big|_0^1 =$$

$$= \frac{1}{3}(1-0) - 2(1-0) + 3(1-0) =$$

$$= \frac{1}{3} + 1 = \frac{4}{3}$$

$$P_2 = -\int_1^2 (x^2-4x+3)\,dx = -\left(\frac{x^3}{3}\Big|_1^2 - 4\cdot\frac{x^2}{2}\Big|_1^2 + 3x\Big|_1^2\right) = -\left(\frac{1}{3}(8-1) - 2(4-1) + 3\cdot 1\right)$$

$$= -\left(\frac{7}{3} - 6 + 3\right) = -\left(-\frac{2}{3}\right) = \frac{2}{3}$$

$P = \frac{4}{3} + \frac{2}{3} = 2$ tražena površina ravne figure.

3. Izračunati površinu ravne figure kojeg čine parabola $y=x^2-2x+2$ i prava $x+2y-9=0$.

Rj.

prava $x+2y-9=0$

$2y=-x+9$

$y=-\frac{1}{2}x+\frac{9}{2}$

prava prolazi kroz tačke $A(0,\frac{9}{2})$ i $B(9,0)$.

$y=x^2-2x+2$

$D=4-8=-4<0$

kriva ne siječe x-o
 – nema nula

$x=0 \Rightarrow y=2$

$C(0,2)$ je presjek krive sa Y-osom

$$-\frac{b}{2a} = -\frac{-2}{2} = 1$$

$$\frac{4ac-b^2}{4a} = \frac{8-4}{4} = 1.$$

$T(1,1)$ je tjeme parabole

Trebamo naći još tačke presjeka prave i parabole.

$y=x^2-2x+2$

$x+2y-9=0$

$y=x^2-2x+2$

$x=-2y+9$

$y=(-2y+9)^2-2(-2y+9)+2$

$y=4y^2-36y+81+4y-18+2$

$4y^2-33y+65=0$

$D=33^2-16\cdot 65=49$

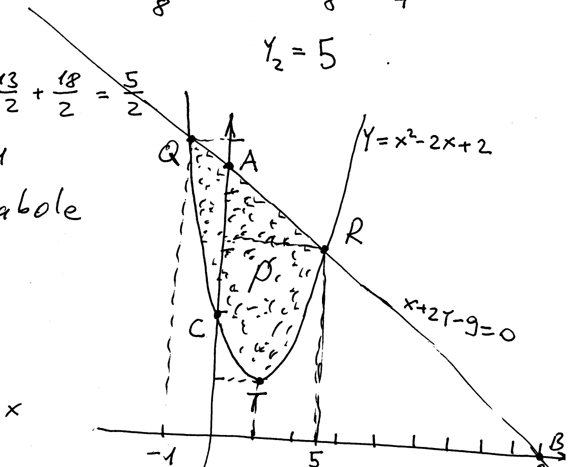
$y_{1,2}=\frac{33\pm 7}{8}$ $\quad y_1=\frac{26}{8}=\frac{13}{4}$

$y_2=5$

$y_1=\frac{13}{4} \Rightarrow x=-2\cdot\frac{13}{4}+9=-\frac{13}{2}+\frac{18}{2}=\frac{5}{2}$

$y_2=5 \Rightarrow x=-2\cdot 5+9=-1$

Tačke presjeka prave i parabole su $R(\frac{5}{2},\frac{13}{4})$ i $Q(-1,5)$



$$P=\int_{-1}^{\frac{5}{2}}\left[\left(-\frac{1}{2}x+\frac{9}{2}\right)-(x^2-2x+2)\right]dx$$

$$\int_{-1}^{\frac{5}{2}}\left(-\frac{1}{2}x+\frac{9}{2}\right)dx = -\frac{1}{2}\cdot\frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}} + \frac{9}{2}x\Big|_{-1}^{\frac{5}{2}} = -\frac{1}{4}\left(\frac{25}{4}-1\right)+\frac{9}{2}\left(\frac{5}{2}-(-1)\right)$$

$$= -\frac{1}{4}\cdot\frac{21}{4}+\frac{9}{2}\cdot\frac{7}{2} = \frac{231}{16}$$

$$\int_{-1}^{\frac{5}{2}}(x^2-2x+2)\,dx = \frac{x^3}{3}\Big|_{-1}^{\frac{5}{2}} - 2\frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}} + 2x\Big|_{-1}^{\frac{5}{2}} = \frac{1}{3}\left(\frac{125}{8}-(-1)\right)-\left(\frac{25}{4}-1\right)+$$

$$+2\left(\frac{5}{2}-(-1)\right) = \frac{1}{3}\cdot\frac{133}{8} - \frac{21}{4} + 2\cdot\frac{7}{2} = \frac{133}{24} + \frac{7}{4} = \frac{175}{24}$$

$$P = \frac{231}{16} - \frac{175}{24} = \frac{231}{4\cdot 4} - \frac{175}{6\cdot 4} = \frac{686}{96} = \frac{343}{48}$$

4. Izračunati površinu ravne figure koja je ograničena krivom $y^2=2x+1$ i pravom $y=2x-1$.

Rj. prava $y=2x-1$ prolazi kroz tačke $A(0,-1)$ i $B(\frac{1}{2},0)$.

$y^2 = 2x + 1$

$2x = y^2 - 1$

$x = \frac{1}{2}y^2 - \frac{1}{2}$

Kriva oblika $x = ay^2 + bx + c$ ima grafik u obliku parabole

ili

$a < 0$ $a > 0$

Tjeme krive se traži po formuli $T\left(-\frac{D}{4a}, -\frac{b}{2a}\right)$

$x = 0 \Rightarrow y^2 = 1$

$A(0,-1)$ i $B(0,1)$ su tačke presjeka f-je sa y-osom

$C(-\frac{1}{2}, 0)$ je nula f-je

$D = 1 \qquad -\frac{D}{4a} = -\frac{1}{4\cdot\frac{1}{2}}$

$-\frac{b}{2a} = -\frac{0}{2\cdot\frac{1}{2}} = 0$

$T(-\frac{1}{2}, 0)$ je tjeme parabole

Tražimo još tačke presjeka krive i prave

$y = 2x - 1$

$y^2 = 2x + 1$

za $x = 0$ $\Downarrow$ $y = -1$

$(2x-1)^2 = 2x + 1$

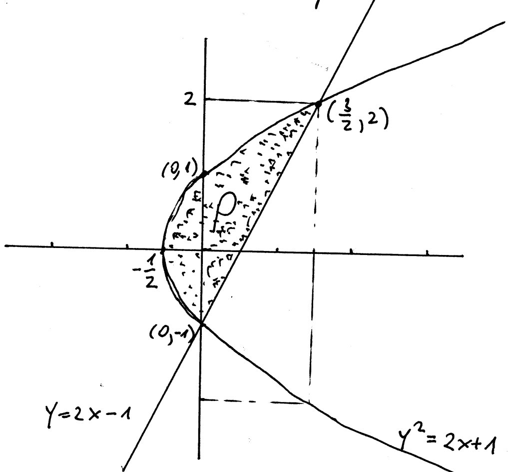
$4x^2 - 4x + 1 - 2x - 1 = 0$

$4x^2 - 6x = 0$

$2x(2x - 3) = 0$

$x = 0 \lor x = \frac{3}{2}$

$D(0,-1)$ i $E(\frac{3}{2}, 2)$ su tačke presjeka krive i prave



$y = 2x - 1 \Rightarrow x = \frac{1}{2}y + \frac{1}{2}$

$y^2 = 2x + 1 \Rightarrow x = \frac{1}{2}y^2 - \frac{1}{2}$

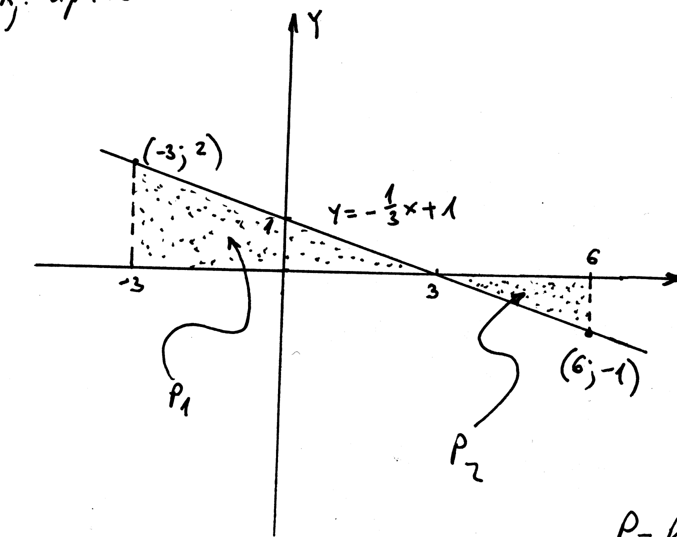
$$P = \int_{-1}^{2}\left[\frac{1}{2}y + \frac{1}{2} - \left(\frac{1}{2}y^2 - \frac{1}{2}\right)\right]dy = \frac{1}{2}\int_{-1}^{2}(y + 1 - y^2 + 1)\,dy = \frac{1}{2}\int_{-1}^{2}(-y^2 + y + 2)\,dy =$$

$$= \frac{1}{2}\cdot\left[\left(-\frac{y^3}{3}\right)\Big|_{-1}^{2} + \frac{y^2}{2}\Big|_{-1}^{2} + 2y\Big|_{-1}^{2}\right] = \frac{1}{2}\left[-\frac{1}{3}(8+1) + \frac{1}{2}(4-1) + 2(2+1)\right] =$$

$$= \frac{1}{2}\left(-3 + \frac{3}{2} + 6\right) = \frac{1}{2}\cdot\frac{9}{2} = \frac{9}{4}$$ tražena površina

Primjenom određenog integrala odrediti površinu figure koju ograničava x-osa zajedno sa linijama $x + 3y - 3 = 0$, $x = -3$ i $x = 6$.

Rj.-uputе



$x + 3y - 3 = 0$

$-3y = x - 3 \quad /:(-3)$

$y = -\frac{1}{3}x + 1$

$P = P_1 + P_2$

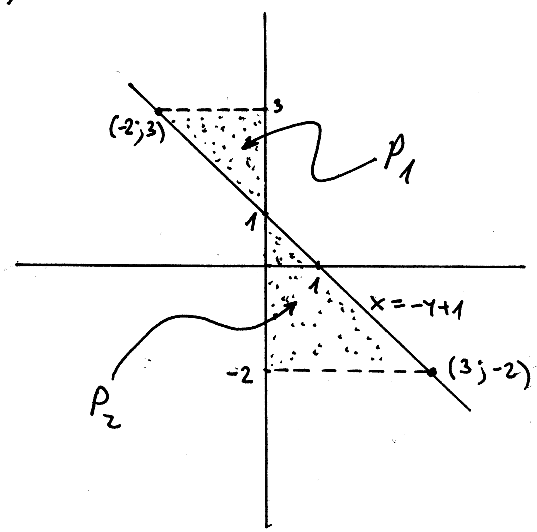
$$P_1 = \int_{-3}^{3}\left(-\frac{1}{3}x + 1\right)dx = \ldots = 6$$

$$P_2 = \left|\int_{3}^{6}\left(-\frac{1}{3}x + 1\right)dx\right| = \ldots = +\frac{3}{2}$$

$$P = 6 + \frac{3}{2} = \frac{15}{2}$$

Primjenom određenog integrala odrediti površinu figure koju ograničava y-osa zajedno sa linijama $x+y-1=0$, $y=3$ i $y=-2$.

Rj.-upute



$$x+y-1=0$$

$$x=-y+1$$

$$P=P_1+P_2$$

$$P_1=\left|\int_1^3(-y+1)\,dy\right|=\ldots=2$$

$$P_2=\int_{-2}^{1}(-y+1)\,dy=\ldots=\frac{9}{2}$$

$$P=2+\frac{9}{2}=\frac{13}{2}$$

Na parabolu $y=1-x^2$ povučena je normala u tački presjeka parabole i pozitivnog dijela x-ose. Odrediti površinu figure koju čine data parabola, povučena normala i y-osa.

Rj.

$y=1-x^2$

$y(0)=1$

$(0,1)$ je presjek sa y-osom

$1-x^2=0$

$x^2=1$

$x_{1,2}=\pm 1$

$(-1,0)$ i $(1,0)$ su nule f-je

$y=-x^2+1$ parabola izgleda

$T\left(-\frac{b}{2a},-\frac{D}{4a}\right)$

$-\frac{b}{2a}=-\frac{0}{2\cdot(-1)}=0$

$D=0-4(-1)(1)=4$

$-\frac{D}{4a}=-\frac{4}{4\cdot(-1)}=1$

$T(0,1)$

$y-y_1=y'(x_1)(x-x_1)$ jednačina tangente u tački (x_1,y_1)

$y-y_1=-\frac{1}{y'(x_1)}(x-x_1)$ jednačina normale u tački (x_1,y_1)

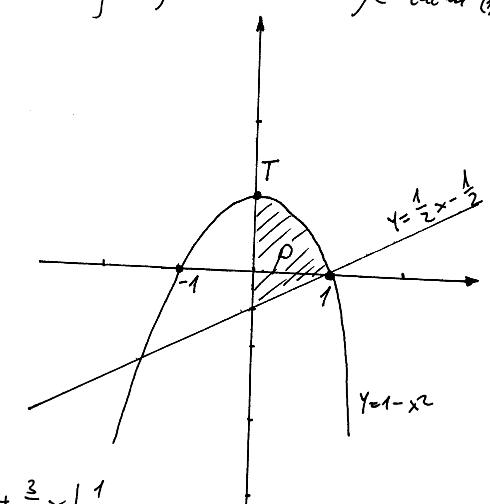
$y'=-2x$

presjek parabole i pozitivnog dijela x-ose je tačka $(1,0)$

$y'(1)=-2$

$y-0=-\frac{1}{-2}(x-1)$

$y=\frac{1}{2}x-\frac{1}{2}$ jednačina normale u tački $(1,0)$



$$P=\int_0^1\left[(1-x^2)-\left(\tfrac{1}{2}x-\tfrac{1}{2}\right)\right]dx=$$

$$=\int_0^1\left(-x^2-\tfrac{1}{2}x+\tfrac{3}{2}\right)dx=-\frac{1}{3}x^3\Big|_0^1-\frac{1}{4}x^2\Big|_0^1+\frac{3}{2}x\Big|_0^1$$

$$=-\frac{1\cdot 4}{3\cdot 4}-\frac{1\cdot 3}{4\cdot 3}+\frac{3}{2}=\frac{3\cdot 6}{2\cdot 6}-\frac{7}{12}=\frac{18-7}{12}=\frac{11}{12}$$

$P=\frac{11}{12}$ tražena površina

Izračunati površinu koju gradi kriva $y=x^2+x-6$ zajedno sa svojim tangentama povučenim na tu krivu u nultačkama krive.

Rj. $y=x^2+x-6$

$D=1+24=25$

$y=(x-2)(x+3)$

$x_1=2 \quad x_2=-3$

$(2,0)$ i $(-3,0)$ su nule f-je

$f(0)=-6$

$(0,-6)$ je tačka presjeka f-je sa y-osom

$y'=2x+1$

$(-3,0),\ y'(-3)=-5$

$y-0=-5(x+3)$

$y=-5x-15$ jednačina tangente na krivu y u tački $(-3,0)$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ je tjeme f-je

$a>0$ f-ja je $\cup$ oblika

$-\frac{b}{2a}=-\frac{1}{2}, \quad -\frac{D}{4a}=-\frac{25}{4}=-6\frac{1}{4}$

$T\left(-\frac{1}{2}, -6\frac{1}{4}\right)$

$y-y_1=k(x-x_1)$ jednačina prave kroz tačku (x_1,y_1) i koeficijentom k

u slučaju tangente $k=y'(x_1)$

$(2,0),\ y'(2)=5$

$y-0=5(x-2)$

$y=5x-10$ jednačina tangente na krivu y u tački $(2,0)$

presjek pravih:

$y=-5x-15 \quad (1)$

$y=5x-10 \quad (2)$

$(1)+(2):\ 2y=-25$

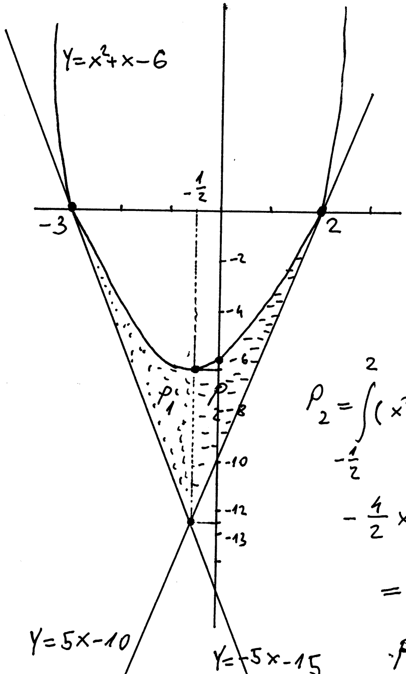
$y=-\frac{25}{2}=-12\frac{1}{2}$

$(1)-(2):\ -10x-5=0$

$-10x=5$

$x=-\frac{1}{2}$

$\left(-\frac{1}{2}, -12\frac{1}{2}\right)$ je tačka presjeka pravih



$P=P_1+P_2$

$$P_1=\int_{-3}^{-\frac{1}{2}}\left(x^2+x-6-(-5x-15)\right)dx=\int_{-3}^{-\frac{1}{2}}(x^2+6x+9)dx=$$

$$=\frac{1}{3}x^3\Big|_{-3}^{-\frac{1}{2}}+\frac{6}{2}x^2\Big|_{-3}^{-\frac{1}{2}}+9x\Big|_{-3}^{-\frac{1}{2}}=\frac{1}{3}\left(-\frac{1}{8}+27\right)$$

$$+3\left(\frac{1}{4}-9\right)+9\left(-\frac{1}{2}+3\right)=\frac{1}{3}\cdot\frac{215}{8}+3\cdot\frac{-35}{4}$$

$$+9\cdot\frac{5}{2}=\frac{215}{24}-\frac{630}{24}+\frac{540}{24}=\frac{125}{24}$$

$$P_2=\int_{-\frac{1}{2}}^{2}\left(x^2+x-6-(5x-10)\right)dx=\int_{-\frac{1}{2}}^{2}(x^2-4x+4)dx=\frac{1}{3}x^3\Big|_{-\frac{1}{2}}^{2}-$$

$$-\frac{4}{2}x^2\Big|_{-\frac{1}{2}}^{2}+4x\Big|_{-\frac{1}{2}}^{2}=\frac{1}{3}\left(8+\frac{1}{8}\right)-2\left(4-\frac{1}{4}\right)+4\left(2+\frac{1}{2}\right)=$$

$$=\frac{1}{3}\cdot\frac{65}{8}-2\cdot\frac{15}{4}+4\cdot\frac{5}{2}=\frac{65}{24}-\frac{180}{24}+\frac{240}{24}=\frac{125}{24}$$

$P=P_1+P_2=\frac{125}{24}+\frac{125}{24}=\frac{125}{12}$ tražena površina

Izračunati površinu figure koju ograničavaju linije $x=y^2-2y-3$ i $y=3-3x$

$y=3-3x$

$3x=3-y$

$x=1-\frac{1}{3}y$

Rj. Nađimo presječnu tačku ovih linija

$x=y^2-2y-3$

$y=3-3x$

$x=(3-3x)^2-2(3-3x)-3$

$x=9-18x+9x^2-6+6x-3$

$9x^2-13x=0$

$x(9x-13)=0$

$x=0$ ili $9x=13$

$x=\frac{13}{9}$

$x=0 \Rightarrow y=3$

$x=\frac{13}{9} \Rightarrow y=3-3\cdot\frac{13}{9}=\frac{9}{3}-\frac{13}{3}=-\frac{4}{3}$

$A(0,3)$ i $B\left(\frac{13}{9},-\frac{4}{3}\right)$ su presječne tačke linija

$x=y^2-2y-3$ je kriva oblika parabole $\subset$ čije je tjeme $T\left(-\frac{D}{4a}, -\frac{b}{2a}\right)$

$-\frac{b}{2a}=-\frac{-2}{2}=1, \quad D=4+12=16, \quad -\frac{D}{4a}=-\frac{16}{4}=-4$

$T(1,-4)$

$y_{1,2}=\frac{2\pm4}{2}$

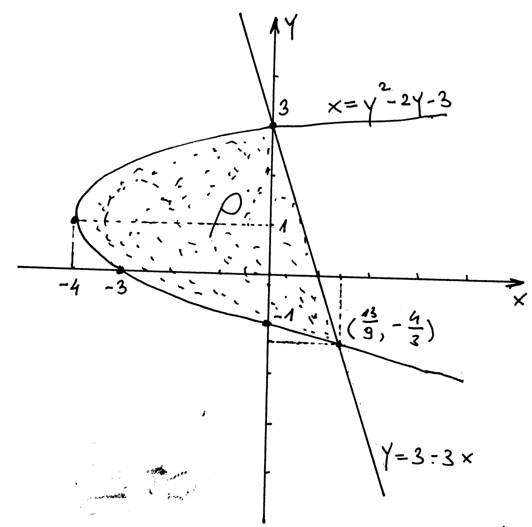
$y_1=\frac{-2}{2}=-1 \quad y_2=\frac{6}{2}=3$

$M_1(0,-1)$ i $M_2(0,3)$ su presjek parabole sa y-osom

$x=y^2-2y-3$

$y=0 \Rightarrow x=-3$

$(-3,0)$ je presjek krive sa x-osom



$$P=\int_{-\frac{4}{3}}^{3}\left[\left(1-\frac{1}{3}y\right)-(y^2-2y-3)\right]dy=$$

$$=\int_{-\frac{4}{3}}^{3}\left(-y^2+\frac{5}{3}y+4\right)dy=$$

$$=-\frac{1}{3}y^3\Big|_{-\frac{4}{3}}^{3}+\frac{5}{3}\cdot\frac{1}{2}y^2\Big|_{-\frac{4}{3}}^{3}+4y\Big|_{-\frac{4}{3}}^{3}=$$

$$=-\frac{1}{3}\left(27+\frac{64}{27}\right)+\frac{5}{6}\left(9-\frac{16}{9}\right)+4\left(3+\frac{4}{3}\right)$$

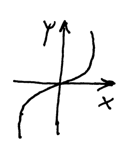
$$=-\frac{1}{3}\cdot\frac{793}{27}+\frac{5}{6}\cdot\frac{65}{9}+4\cdot\frac{13}{3}=$$

$$=-\frac{793}{81}+\frac{325}{54}+\frac{52}{3}=\frac{-793\cdot2+325\cdot3+52\cdot54}{162}=\frac{-1586+975+2808}{162}$$

$P=\frac{2197}{162}=13\frac{91}{162}$ tražena površina

Izračunati površinu figure koja je određena linijama $y=-x$, $y=\sqrt[3]{x}$, $y=3x-2$.

Rj. Grafički nije teško predstaviti prave $y=-x$ i $y=3x-2$.
Problem predstavlja kriva $y=\sqrt[3]{x}$.
Ako znamo da kriva $y=x^3$ izgleda ovako



Onda nije teško nacrtati krivu $x=y^3$ što je ekvivalentno sa $y=\sqrt[3]{x}$.
Pronađimo tačke presjeka datih krivih.

$$\begin{aligned} y&=-x\\ y&=3x-2\\ \hline -x&=3x-2\\ -4x&=-2\\ x&=\tfrac{1}{2} \Rightarrow y=-\tfrac{1}{2}\end{aligned}$$

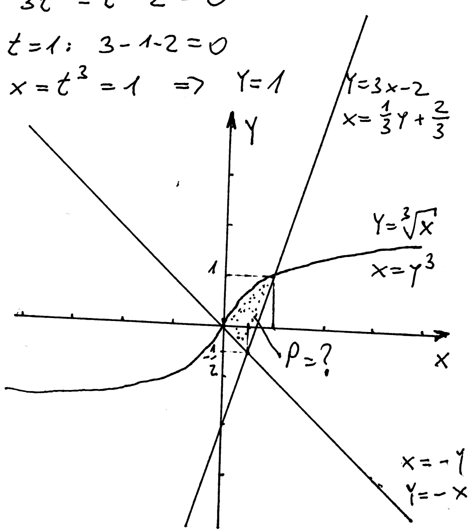
$$\begin{aligned} y&=-x\\ y&=\sqrt[3]{x}\\ \hline y&=-x\\ y^3&=x\\ \hline -x^3&=x\\ x^3+x&=0\\ x(x^2+1)&=0\\ x=0 &\Rightarrow y=0\end{aligned}$$

$$\begin{aligned} y&=3x-2\\ y&=\sqrt[3]{x}\\ \hline \sqrt[3]{x}&=3x-2\\ (3x-2)^3&=x\\ 27x^3-3\cdot(3x)^2\cdot 2&+3\cdot 3x\cdot(-2)^2+(-2)^3=x\\ 27x^3-54x^2+36x-8&=x\\ 27x^3-54x^2+35x-8&=0\end{aligned}$$

pokušajmo riješiti sistem na drugi način

$$\sqrt[3]{x}=3x-2$$
$$x=t^3$$
$$3t^3-2=t$$
$$3t^3-t-2=0$$
$$t=1:\ 3-1-2=0$$
$$x=t^3=1 \Rightarrow y=1$$

$$\boxed{3x=y+2}$$



$$P=\int_{-\frac{1}{2}}^{0}\left[\left(\tfrac{1}{3}y+\tfrac{2}{3}\right)-(-y)\right]dy+\int_{0}^{1}\left[\left(\tfrac{1}{3}y+\tfrac{2}{3}\right)-y^3\right]dy=$$

$$=\int_{-\frac{1}{2}}^{0}\left(\tfrac{4}{3}y+\tfrac{2}{3}\right)dy+\int_{0}^{1}\left(-y^3+\tfrac{1}{3}y+\tfrac{2}{3}\right)dy=$$

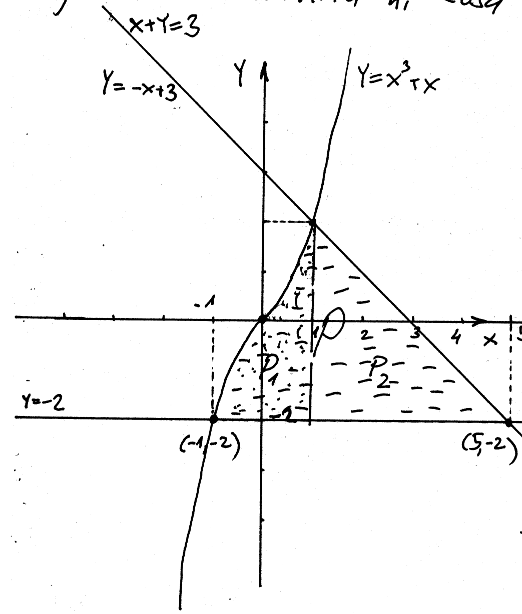
$$=\tfrac{4}{3}\cdot\tfrac{1}{2}y^2\Big|_{-\frac{1}{2}}^{0}+\tfrac{2}{3}y\Big|_{-\frac{1}{2}}^{0}-\tfrac{1}{4}y^4\Big|_{0}^{1}+\tfrac{1}{3}\cdot\tfrac{1}{2}y^2\Big|_{0}^{1}$$

$$+\tfrac{2}{3}y\Big|_{0}^{1}=\tfrac{2}{3}\cdot\left(-\tfrac{1}{\cancel{4}_{2}}\right)+\tfrac{\cancel{2}}{3}\cdot\tfrac{1}{\cancel{2}}-\tfrac{1}{4}+\tfrac{1}{6}+$$

$$+\tfrac{2}{3}=-\tfrac{\cancel{1}}{6}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{\cancel{1}}{6}+\tfrac{2}{3}=\tfrac{3}{4}$$

Izračunati površinu figure koja je određena linijama $y=-2$, $y=x^3+x$, $x+y=3$.

Rj. $y=-2$, $x+y=3$ su prave linije i njih nije teško nacrtati.
Problem za crtanje predstavlja kriva $y=x^3+x$.
Ispitajmo f-ju $y=x^3+x$. $D: x\in\mathbb{R}$
$f(-x)=-x^3-x=-(x^3+x)$ f-ja je neparna
$A(0,0)$ je nula f-je i presjek sa y-osom
f-ja nema prekida $\Rightarrow$ f-ja nema vertikalnu asimptotu
f-ja nema horizontalnu ni kosu asimptotu

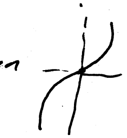


$y'=3x^2+1$ f-ja je uvijek pozitivna (raste za svako x)
f-ja nema ekstrem
$y''=6x$

x	$(0,+\infty)$
y''	$+$
y	$\cup$

$(0,0)$ je prevojna tačka

f-ja je ovog oblika



Nađimo tačke presjeka datih krivih.

$$\begin{aligned} y&=-2\\ x+y&=3\\ \hline x-2&=3\\ x&=5\end{aligned}$$

$(5,-2)$ je tačka presjeka

$$\begin{aligned} y&=-2\\ y&=x^3+x\\ \hline -2&=x^3+x\\ x^3+x+2&=0\end{aligned}$$

$x=-1:\ -1-1+2=0$

$$\begin{aligned}(x^3+x+2)&:(x+1)=x^2-x+2\\ -\ \ x^3+x^2&\\ \hline -x^2+x+2&\\ -\ \ -x^2-x&\\ \hline 2x+2&\\ 2x+2&\\ \hline //&\end{aligned}$$

$$x^3+x+2=(x+1)\underbrace{(x^2-x+2)}_{>0\ \forall x}$$

Rješenje jednačine $x^3+x+2=0$ je $x=-1$.
$(-1,-2)$ je tačka presjeka datih krivih

$$\begin{array}{l} y = x^3 + x \\ x + y = 3 \\ \hline y = x^3 + x \\ y = -x + 3 \\ \hline \end{array}$$

$$-x + 3 = x^3 + x$$

$$x^3 + 2x - 3 = 0$$

$x = 1$: $1^3 + 2\cdot 1 - 3 = 3 - 3 = 0$

$$(x^3 + 2x - 3) : (x - 1) = x^2 + x + 3$$

$$\begin{array}{l} x^3 - x^2 \\ \hline x^2 + 2x - 3 \\ -\ x^2 - x \\ \hline 3x - 3 \\ -\ 3x - 3 \\ \hline = \quad = \end{array}$$

$$x^3 + 2x - 3 = \underbrace{(x^2 + x + 3)}_{>0\ \forall x}(x - 1)$$

$(1, 2)$ je presječna tačka krivih

$$P_1 = \int_{-1}^{1} \left[(x^3 + x) - (-2) \right] dx = \int_{-1}^{1} (x^3 + x + 2)\, dx = \frac{1}{4} x^4 \Big|_{-1}^{1} + \frac{1}{2} x^2 \Big|_{-1}^{1} + 2x \Big|_{-1}^{1} = 4$$

$$P_2 = \int_{1}^{5} \left[(-x + 3) - (-2) \right] dx = \int_{1}^{5} (-x + 5)\, dx = -\frac{x^2}{2} \Big|_{1}^{5} + 5x \Big|_{1}^{5} =$$

$$= -\frac{1}{2}(25 - 1) + 5 \cdot 4 = -\frac{1}{2} \cdot 24 + 20 = 20 - 12 = 8$$

$P = P_1 + P_2 = 8 + 4 = 12$ površina figure

Izračunati površinu figure koju čine linije

$$y = (x-1)^2, \quad \frac{x^2}{1} - \frac{y^2}{2} = 1.$$

Rj. Da bi odredili granice za računanje površine potrebno je grafički predstaviti ove dvije linije.

ispitajmo f-ju $y = (x-1)^2$

D: $x \in \mathbb{R}$

f-ja nije ni parna ni neparna

$f(0) = 1$, $(0,1)$ je presjek sa y-osom

$(x-1)^2 = 0 \Rightarrow x = 1$ $(1,0)$ je nula f-je

$y = (x-1)^2 = x^2 - 2x + 1 \Rightarrow$ f-ja je oblika

Nađimo još tjeme f-je

$y' = 2x - 2$

$y' = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$

$T(1,0)$

Prije nego grafički predstavimo liniju $y = (x-1)^2$ pronađimo u kojim tačkama siječe liniju $\frac{x^2}{1} - \frac{y^2}{2} = 1$.

$$\begin{array}{l} y = x^2 - 2x + 1 = (x-1)^2 \\ 2x^2 - y^2 = 2 \\ \hline \end{array}$$

$$y^2 = (x^2 - 2x + 1)^2 = (x^2 - 2x + 1)(x^2 - 2x + 1) =$$

$$= x^4 - 2x^3 + x^2 - 2x^3 + 4x^2 - 2x + x^2 - 2x + 1$$

$$= x^4 - 4x^3 + 6x^2 - 4x + 1.$$

$$\begin{array}{l} 2x^2 - y^2 = 2 \\ \hline 2x^2 - x^4 + 4x^3 - 6x^2 + 4x - 1 - 2 = 0 \quad / \cdot (-1) \\ x^4 - 4x^3 + 4x^2 - 4x + 3 = 0 \end{array}$$

Označimo ovo sa $g(x) = x^4 - 4x^3 + 4x^2 - 4x + 3$

Kako je $g(1) = 0 \Rightarrow g(x)$ je djeljivo sa $(x-1)$

$$(x^4 - 4x^3 + 4x^2 - 4x + 3) : (x-1) = x^3 - 3x^2 + x - 3$$

$$\begin{array}{l} x^4 - x^3 \\ \hline -3x^3 + 4x^2 - 4x + 3 \\ -\ -3x^3 + 3x^2 \\ \hline x^2 - 4x + 3 \\ -\ x^2 - x \\ \hline -3x + 3 \\ -3x + 3 \\ \hline = \quad = \end{array}$$

$$g(x) = \underbrace{(x^3 - 3x^2 + x - 3)}_{g_1(x)}(x-1)$$

$g_1(0) = -3$

$g_1(1) = 1 - 3 + 1 - 3 = -4$

$g_1(2) = 8 - 12 + 2 - 3 = -5$

$g_1(3) = 27 - 27 + 3 - 3$

$g_1(-2) = -8 - 12 - 2 - 3 = -25$

$g_1(-1) = -1 - 3 - 1 - 3 = -8$

$g_1(-3) = -27 - 27 - 3 - 3 < 0$

$\Rightarrow g_1(x)$ je djeljivo sa $x - 3$

$$(x^3 - 3x^2 + x - 3) : (x - 3) = x^2 + 1$$

$$\begin{array}{l} -\ x^3 - 3x^2 \\ \hline x - 3 \\ -\ x - 3 \\ \hline = \quad = \end{array}$$

Prema tome $g(x) = (x^2 + 1)(x - 3)(x - 1)$

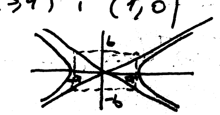
Za $x = 3 \Rightarrow y = 4$

Za $x = 1 \Rightarrow y = 0$

Presječne tačke krivih su $(3,4)$ i $(1,0)$

Krive oblika

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ zovemo HIPERBOLE i one su oblika

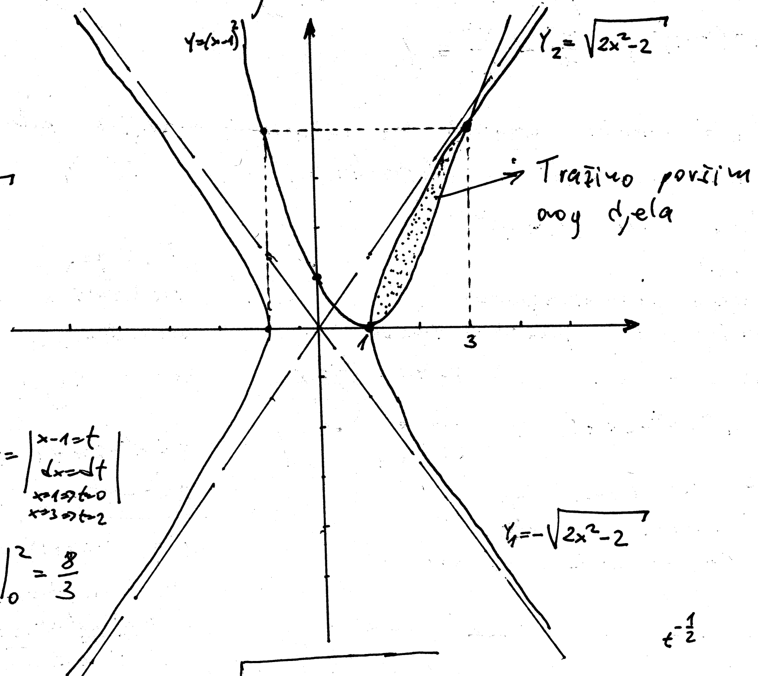


Nacrtajmo naše krive linije

$\sqrt{2} \approx 1,41$

$y^2 = 2x^2-2$

$Y_{1,2} = \pm\sqrt{2x^2-2}$



$$P_2=\int_1^3 (x-1)^2\,dx = \left| \begin{matrix} x-1=t\\ dx=dt \\ x=1\Rightarrow t=0\\ x=3\Rightarrow t=2\end{matrix}\right| = \int_0^2 t^2\,dt = \frac{1}{3} t^3\Big|_0^2 = \frac{8}{3}$$

$t^{-\frac{1}{2}}$

$$\boxed{\int \frac{x}{\sqrt{x^2-1}}\,dx = \left|\begin{matrix}x^2-1=t\\2x\,dx=dt\\x\,dx=\frac{1}{2}dt\end{matrix}\right| = \frac{1}{2}\int\frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+C = \sqrt{t}+C = \sqrt{x^2-1}+C}$$

$$P=\int_1^3\Big(\underbrace{\sqrt{2x^2-2}}_{P_1} - \underbrace{(x-1)^2}_{P_2}\Big)\,dx$$

$$P_1=\int_1^3 \sqrt{2}\cdot\sqrt{x^2-1}\,dx = \sqrt{2}\int_1^3\frac{x^2-1}{\sqrt{x^2-1}}\,dx = \sqrt{2}\left(\int_1^3\frac{x^2}{\sqrt{x^2-1}}\,dx - \int_1^3\frac{dx}{\sqrt{x^2-1}}\right)$$

$$\int_1^3 x\cdot\frac{x}{\sqrt{x^2-1}}\,dx = \left|\begin{matrix}u=x & dv=\frac{x}{\sqrt{x^2-1}}dx\\ du=dx & v=\sqrt{x^2-1}\end{matrix}\right| = \underbrace{x\sqrt{x^2-1}\Big|_1^3}_{3\sqrt{8}-0} - \int_1^3\sqrt{x^2-1}\,dx$$

$$\int_1^3\frac{dx}{\sqrt{x^2-1}} = \ln|x+\sqrt{x^2-1}|\Big|_1^3 = \ln|3+\sqrt{8}|$$

$$\sqrt{2}\int_1^3\sqrt{x^2-1}\,dx = \sqrt{2}\cdot\underbrace{3\sqrt{8}}_{6\sqrt{2}} - \sqrt{2}\int_1^3\sqrt{x^2-1}\,dx - \sqrt{2}\ln(3+2\sqrt{2})$$

$$\int_1^3\sqrt{2x^2-2}\,dx = 6 - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

$$P=P_1-P_2 = \frac{10}{3} - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

tražena površina

Dio tablice integrala

1. $\int u^a\,du = \frac{u^{a+1}}{a+1}+C,\quad a\neq-1.$
2. $\int u^{-1}\,du = \int\frac{du}{u} = \int\frac{u'}{u}\,dx = \ln|u|+C.$
3. $\int a^u\,du = \frac{a^u}{\ln a}+C;\ \int e^u\,du = e^u+C.$
4. $\int \sin u\,du = -\cos u + C.$
5. $\int \cos u\,du = \sin u + C.$
6. $\int \sec^2 u\,du = \operatorname{tg} u + C.$
7. $\int \operatorname{cosec}^2 u\,du = -\operatorname{ctg} u + C.$
8. $\int \frac{du}{u^2+a^2} = \frac{1}{a}\operatorname{arc\,tg}\frac{u}{a} + C.$
9. $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C.$
10. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C.$
11. $\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right| + C.$

Diferencijalne jednačine

Osnovni pojmovi

Jednačinu oblika $F(x,y,y')=0$ nazivamo diferencijalnom jednačinom prvog reda. Npr. diferencijalne jednačine prvog reda su:

a) $x^5 - yy' + xy' - 7 = 0$ (u ovom slučaju $F(x,y,y') = x^5 - yy' + xy' - 7$)

b) $y - yxy' = 0$, (u ovom slučaju $F(x,y,y') = y - yxy'$)

c) $y' = 7$

d) $xy' + 1 = 0$

Jednačinu oblika $F(x,y,y',y'')=0$ nazivamo diferencijalnom jednačinom drugog reda. Npr. $y'' - xy = 4x^2$

$F(x,y,y',\dots,y^{(n)})=0$ diferencijalna jednačina n-tog reda

Npr. a) $y^{IV} - 3y'' + 3 = x^2$ diferencijalna jednačina 4-og reda

b) $x^8 - y^{(9)} + y'' = 1$ dif. jedn. devetog reda

1. Provjeriti da li su navedene f-je rješenja datih diferencijalnih jednačina.

a) diferencijalna jednačina $xy' = 2y$, f-ja $y = 5x^2$

b) diferencijalna jednačina $y'' + y = 0$, f-ja $y = \sin x$

c) $y'' = x^2 + y^2$, $y = \frac{1}{x}$

d) $y'' + y = 0$, $y = 3\sin x - 4\cos x$

Rj. a) $xy' = 2y$, $y = 5x^2$, $y' = 10x$

$x \cdot y' = x \cdot 10x = 10x^2 \quad \dots (1)$

$2y = 10x^2 \quad \dots (2)$

(1) i $(2) \Rightarrow xy' = 2y$ F-ja $y = 5x^2$ jest rješenje diferencijalne jednačine $xy' = 2y$

b) $y'' + y = 0$, $y = \sin x$

$y' = \cos x$

$y'' = -\sin x$

$y'' + y = -\sin x + \sin x = 0$

$y'' + y = 0$

F-ja $y = \sin x$ jest rješenje dif. jedn. $y'' + y = 0$.

c) $y'' = x^2 + y^2$, $y = \frac{1}{x} = x^{-1}$

$y' = (-1)x^{-2}$

$y'' = 2x^{-3}$

$y'' = x^2 + y^2$ $\quad (y^2 = (x^{-1})^2 = x^{-2})$

$2x^{-3} = x^2 + x^{-2} \quad / \cdot x^3 \ (x \neq 0)$

$2 = x^5 + x$

F-ja $y = \frac{1}{x}$ nije rješenje diferencijalne jednačine $y'' = x^2 + y^2$.

d) URADITI ZA VJEŽBU Rj. jest

2. Odrediti diferencijalne jednačine prvog reda čija su rješenja: (C, C_1, C_2 su konstante)

a) $y = Cx$ b) $y^2 = 2Cx$ c) $y = C_1(x - C_2)^2$

Rj. a) $y = Cx \quad /'$

$y' = C$

$y = y'x$

$y - y'x = 0$ je difer. jedn. prvog reda čije je rješenje $y = Cx$.

b) $y^2 = 2Cx \quad /'$

$2yy' = 2C$

$C = yy'$

$y^2 = 2yy'x \quad / : y$

$y = 2y'x$

$y - 2y'x = 0$ je difer. jedn. prvog reda čije je rješenje $y^2 = 2Cx$.

c) URADITI ZA VJEŽBU

rješenje $2yy'' = y'^2$ je difer. jedn. prvog reda čije je rješenje $y = C_1(x - C_2)^2$.

(#) Provjeriti da li je data f-ja rješenje date diferencijalne jednačine

a) $y=\sqrt{x}$, $2yy'=1$

b) $\ln x \ln y = c$, $y\ln y\,dx + x\ln x\,dy = 0$

c) $s=-t-\frac{1}{2}\sin 2t$, $\frac{d^2s}{dt^2}+\operatorname{tg} t\,\frac{ds}{dt}=\sin 2t$.

Rj:

a) $y=\sqrt{x}=(x)^{\frac{1}{2}}$

$$y'=\frac{1}{2\sqrt{x}} \qquad y\cdot y'=\sqrt{x}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2}$$

$$2yy'=1$$

F-ja $y=\sqrt{x}$ je rješenje diferencijalne jednačine $2yy'=1$.

b) $\ln x \ln y = c \quad /d$

$$\frac{\partial(\ln x\ln y)}{\partial x}dx+\frac{\partial(\ln x\ln y)}{\partial y}dy=0$$

$$\ln y\cdot\frac{1}{x}dx+\ln x\cdot\frac{1}{y}dy=0$$

$$\ln x\,\frac{1}{y}dy=-\ln y\cdot\frac{1}{x}dx \quad /\cdot\frac{y}{\ln x}$$

$$dy=-\frac{y\ln y}{x\ln x}dx$$

Uvrštimo dobijeni izraz za dy u jednačinu $y\ln y\,dx+x\ln x\,dy=0$

$$y\ln y\,dx+x\ln x\left(-\frac{y\ln y}{x\ln x}dx\right)=0$$

$$0=0$$

Prema tome f-ja $\ln x\ln y=c$ je rješenje diferencijalne jednačine $y\ln y\,dx+x\ln x\,dy=0$.

c) $s=-t-\frac{1}{2}\sin 2t$

$$\frac{ds}{dt}=-1-\frac{1}{2}\cdot 2\cos 2t=-1-\cos 2t$$

$$\frac{d^2s}{dt^2}=2\sin 2t$$

$$\frac{d^2s}{dt^2}+\operatorname{tg} t\,\frac{ds}{dt}=\sin 2t$$

$$2\sin 2t+\operatorname{tg} t(-1-\cos 2t)=\sin 2t$$

$$\sin 2t+\operatorname{tg} t(\underline{-\sin^2 t}-\cos^2 t-\cos^2 t\underline{+\sin^2 t})=0$$

$$\sin 2t-2\cos^2 t\operatorname{tg} t=0$$

$$\sin 2t-2\cos^2 t\,\frac{\sin t}{\cos t}=0$$

$$\sin 2t-2\sin t\cos t=0$$

$$0=0$$

F-ja $s=-t-\frac{1}{2}\sin 2t$ je rješenje diferencijalne jednačine $\frac{d^2s}{dt^2}+\operatorname{tg} t\,\frac{ds}{dt}=\sin 2t$.

(#) Ako znamo opšte rješenje $4x^2+y^2=C^2$ neke diferencijalne jednačine prvog reda, odrediti i grafički prikazati integralne krive (parcijalni integrali) koje prolaze kroz tačke $B_1(-1;0)$, $B_2(0;-3)$ i $B_3(2;0)$.

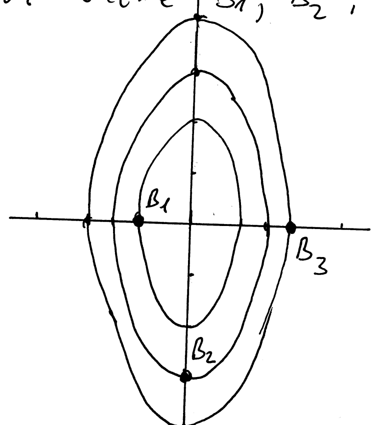
Rj: Opšte rješenje $F(x,y,C)=0$ diferencijalne jednačine prvog reda $f(x,y,y')=0$ geometrijski definišu familiju krivih koje zavise samo od parametra C. Zamjenjujući u opšte rješenje koordinate tačke P, odredit ćemo vrijednost C, u kojoj opšte rješenje integralne krive, prolazi kroz tačku P.

Za tačku B_1: $4=C^2$; $4x^2+y^2=4$.

Za tačku B_2: $9=C^2$; $4x^2+y^2=9$.

Za tačku B_3: $16=C^2$; $4x^2+y^2=16$.

Za dobijene jednakosti integralne krive prolaze kroz tačke B_1, B_2 i B_3. Nacrtajmo ove krive.



Date krive su koncentrične elipse čiji je centar u koordinatnom početku

Zadaci za vježbu

Provjeriti da li su date f-je rješenja datih jednačina:

1. $y=Ce^{-2x}$; $y'+2y=0$.
2. $y=C_1x+C_2x^2$; $x^2y''-2xy'+2y=0$.
3. $x^2+2xy=C$; $(x+y)\,dx+x\,dy=0$.
4. $s=t^2\ln t+C_1t^2+C_2t+C_3$; $t\dfrac{d^3s}{dt^3}=2$.
5. * $y-x+C_1\ln y=C_2$; $yy''-(y')^2+(y')^3=0$.

Diferencijalne jednačine prvog reda

Diferencijalne jednačine prvog reda sa:

a) diferencijalne jednačine sa razdvojenim promjenjivim

$$y' = f(x)\, g(y)$$

npr. $y' = -x \cdot \frac{1}{y}$, $y' = -x\, \frac{1}{y} e^{y}$,

$$\frac{tg\, x}{\cos^2 x}\, dx + \frac{\sin^2}{ctg\, y}\, dy = 0 , \quad \left(y' = \frac{dy}{dx}\right)$$

b) homogene diferencijalne jednačine

$y' = f\left(\frac{y}{x}\right)$, uvodimo smjenu $\frac{y}{x} = u \Rightarrow y = ux$, $y' = u'x + u$

npr. $y' = \dfrac{1 + \left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^3}$, $y' = \frac{y}{x} + \left(1 + \frac{y}{x}\right) \ln\left(1 + \frac{y}{x}\right)$

c) diferencijalne jednačine koje se svode na homogene

$y' = f\left(\dfrac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right)$, ako je $a_1 b_2 - b_1 a_2 = 0$ uvodimo smjenu $a_1 x + b_1 y = u$ i dobijemo diferenc. jedn. sa razdvojenim promjenjivim

ako je $a_1 b_2 - b_1 a_2 \neq 0$ uvodimo smjenu $x = u + \alpha$, $y = v + \beta$ gdje α i β dobijemo iz sistema $a_1 \alpha + b_1 \beta + c_1 = 0$, $a_2 \alpha + b_2 \beta + c_2 = 0$.

npr. $y' = \dfrac{3y - 7x + 7}{3x - 7y - 3}$

$$(x + y - 3)\, dy = -(2x - 4y + 6)\, dx$$

d) linearne diferencijalne jednačine

$y' + p(x) y = g(x)$, uvodimo smjenu $y = u \cdot v$, $y' = u' \cdot v + u \cdot v'$

npr. $y' + y = -x$, $y' + y \cos x = \frac{1}{2} \sin 2x$

e) Bernulijeva diferencijalna jednačina

$y' + p(x) y = g(x)\, y^n$, $n \in \mathbb{R}$, $n \neq 0$; $n \neq 1$.

uvodimo smjenu $y = uv$, $y' = u'v + u \cdot v'$

npr. $y' + y = x y^3$, $x y' + (x+1) y = 3x^2 e^{-x}$,

$$y' - y\, tg\, x + 2 \sin x - 1 = 0$$

f) Lagranžova diferencijalna jednačina

$y = x f(y') + g(y')$, uvodimo smjene $y' = p$, $x = uv$, $dy = p\, dx$

npr. $y = x \cdot 2y' - y'^2$, $y = x \cdot (-y') + 4\sqrt{y'}$,

$$y = x y' - 2 - y'$$

g) Klerova diferencijalna jednačina

$y = x y' + f(y')$, rješavamo ih na isti način kao što rješavamo Lagranžovu diferenc. jedn.

npr. $y = x y' + \sin y'$, $y = x y' + \dfrac{y'^2}{2}$

1. Odrediti tip diferencijalne jednačine:

a) $yy' + xe^{y} = 0$

Rj. $yy' = -xe^{y}$

$y' = -x\frac{1}{y}e^{y}$ diferencijalna jednačina sa razdvojenim promenj.

b) $y + xy' = 4\sqrt{y'}$

Rj. $y = xy' + 4\sqrt{y'}$ Klerova difer. jedn.

c) $y' - y\,tg x + 2\sin x - 1 = 0$

Rj. $y' - tg x \cdot y = 1 - 2\sin x$ linearna difer. jedn.

d) $xy' - y = (x+y)\ln\frac{x+y}{x}$

Rj. $xy' = y + (x+y)\ln(1+\frac{y}{x})$ $/:x$

$y' = \frac{y}{x} + (1+\frac{y}{x})\ln(1+\frac{y}{x})$ homogena difer. jednačina

e) $xy' = y - xy\sin x$

Rj. $xy' = y(1 - x\sin x)$

$y' = y \cdot \frac{1 - x\sin x}{x}$ difer. jedn. sa razdvojenim promenjivim

f) $(x^2+1)y' - xy^2 = xy(x^2y - 1)$

$y' + \frac{x}{x^2+1}y = xy^2$

Rj. $(x^2+1)y' - xy^2 = xy \cdot x^2y - xy$

$(x^2+1)y' + xy = x^2 \cdot xy^2 + xy^2$

$(x^2+1)y' + xy = xy^2(x^2+1)$ $/:(x^2+1)$

Bernulijeva diferenc. jedn.

Početni uslovi

Rješenje oblika $\varphi(x, y, c) = 0$ diferencijalne jednačine $y' = f(x, y)$ zovemo opšte rješenje diferencijalne jednačine. Ako u opštem rješenju konstanta c dobije neku određenu vrijednost, dobijamo partikularno rješenje diferencijalne jednačine.

Naći ono rješenje diferencijalne jednačine $y' = y$ koje zadovoljava uvjete $y = 1$ za $x = 0$.

Rj. $y' = y$

$\frac{dy}{dx} = y$ $/\cdot\frac{dx}{y}$

$\frac{dy}{y} = dx$ $/\int$

$\int\frac{dy}{y} = \int dx$

$\ln|y| = x + c$

$y = e^{x+c_1}$

$y = Ce^{x}$ opšte rješenje

$y(0) = 1$

$Ce^{0} = 1$

$c = 1$

$y = e^{x}$ partikularno rješenje

Naći opšte rješenje diferencijalne jednačine $y' = x^2 - 2$.

Rj. $\frac{dy}{dx} = x^2 - 2$ $/\cdot dx$

$dy = (x^2-2)\,dx$ $/\int$

$\int dy = \int(x^2-2)\,dx$

$y = \frac{x^3}{3} - 2x + c$ opšte rješenje

Diferencijalne jednačine sa razdvojenim promjenjivim

Diferencijalne jednačine prvog reda $P(x,y)\,dx + Q(x,y)\,dy = 0$ nazivaju se diferencijalne jednačine sa razdvojenim promjenjivim ako se f-je P i Q, kada se razlože na faktore, svaki zavisi samo od jedne promjenjive

$$f_1(x)\,f_2(y)\,dx + \varphi_1(x)\,\varphi_2(y)\,dy = 0.$$

U ovom slučaju, dijeljeći jednakost sa $f_2(y)\,\varphi_1(x)$ dobijamo razdvojene promjenjive

$$\frac{f_1(x)}{\varphi_1(x)}\,dx + \frac{\varphi_2(y)}{f_2(y)}\,dy = 0.$$

Poslije razdvajanja varijabli, poslije čega će svaki član u jednakosti zavisiti samo od jedne varijable, opšte rješenje ćemo dobiti tako što ćemo integrirati svaki član posebno

$$\int \frac{f_1(x)}{\varphi_1(x)}\,dx + \int \frac{\varphi_2(y)}{f_2(y)}\,dy = C.$$

\# Odrediti opšte rješenje sljedećih diferencijalnih jednačina:

a) $(x+1)^3\,dy - (y-2)^2\,dx = 0.$ b) $\dfrac{1}{\cos^2 x \cos y}\,dx = -\operatorname{ctg} x \sin y\,dy.$

c) $(\sqrt{xy} + \sqrt{x})\,y' - y = 0.$ d) $2^{x+y} + 3^{x-2y}\,y' = 0.$

a)

$$(x+1)^3\,dy - (y-2)^2\,dx = 0 \quad /:(x+1)^3(y-2)^2$$

$$\frac{dy}{(y-2)^2} - \frac{dx}{(x+1)^3} = 0 \quad \Big/\!\!\int$$

iz oblika $\dfrac{dy}{(y-2)^2} = \dfrac{dx}{(x+1)^3}$ vidimo da je ovo diferencijalna jednačina sa razdvojenim promjenjivim

$$\int \frac{\overbrace{dy}^{d(y-2)}}{(y-2)^2} - \int \frac{\overbrace{dx}^{=d(x+1)}}{(x+1)^3} = C$$

$$\int (y-2)^{-2}\,d(y-2) - \int (x+1)^{-3}\,d(x+1) = C$$

$$-\frac{1}{y-2} + \frac{1}{2(x+1)^2} = C$$

opšte rješenje diferencijalne jednačine

b)

$$\frac{dx}{\cos^2 x \cos y} = -\operatorname{ctg} x \sin y\,dy \quad \Big/\cdot \frac{\cos y}{\operatorname{ctg} x}$$

$$\frac{dx}{\cos^2 x\,\operatorname{ctg} x} = -\cos y \sin y\,dy$$

ovo je diferencijalna jednačina sa razdvojenim promjenjivim

$$\frac{\operatorname{tg} x}{\cos^2 x}\,dx + \sin y \cos y\,dy = 0 \quad \Big/\!\!\int$$

$$\int \operatorname{tg} x\,d(\operatorname{tg} x) + \int \sin y\,d(\sin y) = C_1$$

$$\frac{1}{2}\operatorname{tg}^2 x + \frac{1}{2}\sin^2 y = \frac{1}{2}C$$

$$\operatorname{tg}^2 x + \sin^2 y = C \, .$$ opšte rješenje diferencijalne jednačine

c) $$(\sqrt{xy} + \sqrt{x})\, y' - y = 0$$

$$(\sqrt{y}+1)\sqrt{x}\, y' = y$$

$$y' = \frac{1}{\sqrt{x}} \frac{y}{\sqrt{y}+1}$$ ovo je diferencijalna jednačina sa razdvojenim promjenjivim

$$y' = \frac{dy}{dx} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}}\frac{y}{\sqrt{y}+1}$$

$$(\sqrt{y}+1)\sqrt{x}\frac{dy}{dx} = y \Rightarrow \frac{\sqrt{y}+1}{y}dy = \frac{1}{\sqrt{x}}dx$$

$$\frac{\sqrt{y}+1}{y}dy - \frac{1}{\sqrt{x}}dx = 0 \quad //\int$$

$$\int \left(y^{-1/2} + \frac{1}{y}\right)dy - \int x^{-\frac{1}{2}}dx = C$$

$$2\sqrt{y} + \ln|y| - 2\sqrt{x} = C$$ opšte rješenje diferencijalne jednačine

d) $$2^{x+y} + 3^{x-2y}y' = 0$$

$$2^x \cdot 2^y + 3^x \cdot 3^{-2y}\frac{dy}{dx} = 0 \quad \Big/ \cdot \frac{dx}{2^y 3^x}$$

primjetimo da je riječ o diferencijalnoj jednačini sa razdvojenim promjenjivim

$$\frac{2^x}{3^x}dx + \frac{3^{-2y}}{2^y}dy = 0 \quad //\int$$

$$\int\left(\frac{2}{3}\right)^x dx + \int\left(\frac{1}{18}\right)^y dy = C \qquad \frac{\left(\frac{2}{3}\right)^x}{\ln\frac{2}{3}} - \frac{\left(\frac{1}{18}\right)^y}{\ln 18} = C$$

opšte rješenje diferencijalne jednačine

(#) Odrediti partikularno rješenje diferencijalne jednačine, koji zadovoljavaju inicijalni uslov:

a) $y\,dx + \operatorname{ctg} x\, dy = 0; \quad y\left(\frac{\pi}{3}\right) = -1$

b) $s = s' \cos^2 t \ln s; \quad s(\pi) = 1.$

Rj. a) $$y\,dx + \operatorname{ctg}x\,dy = 0 \quad \Big/ \cdot \frac{1}{y \operatorname{ctg} x}$$

$$\frac{dx}{\operatorname{ctg}x} + \frac{dy}{y} = 0 \quad //\int$$

$$\int \frac{\sin x}{\cos x}dx + \int\frac{dy}{y} = C_1$$

$$-\ln|\cos x| + \ln|y| = \ln C_2$$

$$\ln|y| = \ln C_2|\cos x|$$

$$|y| = C_2|\cos x|$$

$$y = \pm C_2 \cos x = C\cos x$$

$y = C\cos x$ je opšte rješenje diferencijalne jednačine

Da bi odredili partikularno rješenje trebamo odrediti konstantu C tako da je $y\left(\frac{\pi}{3}\right) = -1$. Ovo znači da je $y=-1$, $x = \frac{\pi}{3}$ u opštem rješenju diferencijalne jednačine.

$$-1 = C \cdot \underbrace{\cos\left(\frac{\pi}{3}\right)}_{=\frac{1}{2}} \Rightarrow C = \frac{-1}{\frac{1}{2}} = -2$$

$y = -2\cos x$ je partikularno rješenje diferencijalne jednačine

b) $s = s' \cos^2 t \ln s$

$$s = \frac{ds}{dt} \cos^2 t \ln s \quad \Big/ \frac{dt}{\cos^2 t} \cdot \frac{1}{s}$$

$$\frac{dt}{\cos^2 t} = \frac{\ln s}{s} ds \quad \Big/ \int$$

$$\int d(\operatorname{tg} t) = \int \ln s \, d(\ln s) + C$$

$$\operatorname{tg} t = \frac{1}{2} \ln^2 s + C$$ opšte rješenje diferencijalne jednačine

Sad ako stavimo da je $t = \pi$, $s = 1$ imam

$$\underbrace{\operatorname{tg} \pi}_{=0} = \frac{1}{2} \underbrace{\ln^2 1}_{=0} + C \quad \Rightarrow \quad C = 0$$ je partikularno rješenje diferencijalne jednačine

Zadaci za vježbu

Odrediti opšte rješenje sljedećih diferencijalnih jednačina

1. $(y + xy)\,dx + (x - xy)\,dy = 0$
2. $yy' + x = 1$
3. $\sin\alpha \cos\beta \, d\alpha = \cos\alpha \sin\beta \, d\beta$
4. $1 + (1 + y')\, e^y = 0$
5. $3e^x \sin y \, dx = (e^x - 1) \dfrac{dy}{\cos y}$
6. * $x^2 (2yy' - 1) = 1$

Naći partikularno rješenje sljedećih diferencijalnih jednačina, tako da zadovoljavaju dati inicijalni uslov

7. $y^2 + x^2 y' = 0; \quad y(-1) = 1$
8. $2(1 + e^x)\, yy' = e^x; \quad y(0) = 0.$
9. $(1 + x^2)\, y^3 dx - (y^2 - 1)\, x^3 dy = 0; \quad y(1) = -1.$

Rješenja:

1. $x - y + \ln|xy| = C$ 2. $(x-1)^2 + y^2 = C^2$ 3. $\cos\beta = C \cos\alpha$

4. $(e^y + 1)\, e^x = C$ 5. $\operatorname{tg} y = C (e^x - 1)^3$ 6. $x(y^2 + C) = x^2 - 1$

7. $x + y = 0$ 8. $2e^{y^2} = e^x + 1$ 9. $x^{-2} + y^{-2} = 2\left(1 + \ln\left|\frac{x}{y}\right|\right)$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Diferencijalne jednačine sa razdvojenim promjenjivim)

Diferencijalne jednačine sa razdvojenim promjenjivim su oblika y'=f(x)g(y).

1. Riješiti diferencijalnu jednačinu $xy' = y - xy\sin x$.

Rj. $xy' = y - xy\sin x$

$xy' = y(1 - x\sin x) \quad /: x \ (x \neq 0)$

$y' = y \cdot \dfrac{1 - x\sin x}{x}$ ovo je dif. jedn. sa razdv. promj.

$\dfrac{dy}{dx} = y \cdot \dfrac{1 - x\sin x}{x} \quad / \cdot \dfrac{dx}{y}$

$\dfrac{dy}{y} = \left(\dfrac{1}{x} - \sin x\right) dx$

$\int \dfrac{dy}{y} = \int \dfrac{1}{x} dx - \int \sin x \, dx$

$\ln|y| = \ln|x| + \cos x + \ln C$

$\ln|y| = \ln|x \cdot c| + \ln e^{\cos x}$

$y = Cx e^{\cos x}$ opšte rješenje dif. jedn.

ISPITNI ZADATAK

2. Riješiti diferencijalnu jednačinu

$(xy^2 + 3x)dx + (2x^2y - 5y)dy = 0.$

Rj. $(2x^2y - 5y)dy = -(xy^2 + 3x)dx$

$y(2x^2 - 5)dy = -x(y^2 + 3)dx$

$\dfrac{y}{y^2 + 3} dy = \dfrac{-x}{2x^2 - 5} dx$ ovo je dif. jedn. sa razdv. prom.

$\int \dfrac{y}{y^2+3} dy = -\int \dfrac{x}{2x^2 - 5} dx$

$\int \dfrac{y}{y^2+3} dy = \left| \begin{matrix} y^2 = t \\ 2y\,dy = dt \end{matrix} \right| = \dfrac{1}{2} \int \dfrac{dt}{t+3} = \dfrac{1}{2} \ln|y^2 + 3|$

$\int \dfrac{x}{2x^2-5} dx = \left| \begin{matrix} 2x^2 = t \\ 4x\,dx = dt \\ x\,dx = \frac{1}{4} dt \end{matrix} \right| = \dfrac{1}{4} \int \dfrac{dt}{t-5} = \dfrac{1}{4} \ln|2x^2 - 5|$

$\dfrac{1}{2} \ln|y^2 + 3| = -\dfrac{1}{4} \ln|2x^2 - 5| + \ln C_1 \quad / \cdot 4$

$\ln|y^2+3|^2 = \ln|C \cdot (2x^2 - 5)^{-1}|$

$(y^2 + 3)^2 = \dfrac{C}{2x^2 - 5}$

opšte rješenje dif. jedn.

3. Riješiti diferencijalnu jednačinu

$3y'(x^2 - 1) - 2xy = 0$

Rj. $y^3 = C(x^2 - 1)$ opšte rješ. dif. jedn.

\# Riješiti diferencijalnu jednačinu $y - xy' = a(1 + x^2y')$, $a = const.$

Rj. $y - xy' = a(1 + x^2y')$, $a = const.$

$y - xy' = a + ax^2y'$

$ax^2y' + xy' = y - a$

$(ax^2 + x)y' = y - a$

$y' = \dfrac{1}{ax^2 + x} \cdot (y - a)$ Ovo je diferenc. jednačina sa razdvojenim promjenjivim

$y' = \dfrac{dy}{dx}$

$\dfrac{dy}{y-a} = \dfrac{dx}{ax^2 + x}$

$\int \dfrac{dx}{x(ax+1)} = \int \dfrac{dy}{y-a}$

$\int \dfrac{dx}{x(ax+1)} = \int \left(\dfrac{1}{x} - \dfrac{a}{ax+1}\right) dx$

$ax + 1 = t$

$a\,dx = dt$

$dx = \frac{1}{a} dt$

$= \ln|x| - a \cdot \dfrac{1}{a} \ln|ax+1| + c$

$= \ln\left|\dfrac{x}{ax+1}\right| + c$

$\ln\left|\dfrac{x}{ax+1}\right| = \ln|y - a| + \ln C$

$\dfrac{x}{ax+1} = C(y - a)$ Rešenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu $(x^2y+x^2)dx+(x^4y-y)dy=0$.

Rj: $(x^2y+x^2)dx+(x^4y-y)dy=0$

$x^2(y+1)dx+(x^4-1)y\,dy=0$

$x^2(y+1)dx=-(x^4-1)y\,dy$

$\frac{y}{y+1}dy=-\frac{x^2}{x^4-1}dx$ diferencijalni račun sa razdvojenim promjenjivim //

$$\int\frac{y}{y+1}dy=-\int\frac{x^2}{x^4-1}dx$$

$$\int\frac{y+1-1}{y+1}dy=\int dy-\int\frac{dy}{y+1}=y-\ln|y+1|+c$$

$$\int\frac{x^2}{x^4-1}=\frac{x^2}{(x-1)(x+1)(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x^2+1}\quad /(x^4-1)$$

$$x^2=A\underbrace{(x+1)(x^2+1)}_{x^3+x+x^2+1}+B\underbrace{(x-1)(x^2+1)}_{x^3+x-x^2-1}+C\underbrace{(x-1)(x+1)}_{x^2+x-x-1}$$

$$x^2=A(x^3+x^2+x+1)+B(x^3-x^2+x-1)+C(x^2-1)$$

$A+B=0$ (a) $\qquad A=-B$

$A-B+C=1$ (b) $\qquad$ (b): $-B-B+C=1 \Rightarrow -2B+C=1$

$A+B=0$ (c) $\qquad$ (d): $-B-B-C=0 \Rightarrow -2B-C=0$ $\Big\}+ \Rightarrow -4B=1$, $B=-\frac14$

$A-B-C=0$ (d)

$\Rightarrow A=\frac14 \qquad \frac14+\frac14+C=1 \Rightarrow C=\frac12$

$$\int\frac{x^2}{x^4-1}dx=\frac14\int\frac{dx}{x-1}-\frac14\int\frac{dx}{x+1}+\frac12\int\frac{dx}{x^2+1}=\frac14\ln|x-1|-\frac14|x+1|+\frac12\,arctg\,x+C$$

$$y-\ln|y+1|=\frac14\ln\left|\frac{x-1}{x+1}\right|+\frac12\,arc\,tg\,x+C$$

rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu $y'=2^{2x+y}$.

Rj. $y'=2^{2x}\cdot2^y$ diferencijalna jednačina sa razdvojenim promjenjivim

$\frac{dy}{dx}=2^{2x}\cdot2^y$

$\frac{dy}{2^y}=4^x dx$ $\qquad \int a^x dx=\frac{a^x}{\ln a}+C,\ 0<a\neq1$

$2^{-y}dy=4^x dx$ //

$\int 2^{-y}dy=\int 4^x dx$ $\qquad \ln4=\ln2^2=2\ln2$

$-\frac{2^{-y}}{\ln2}=\frac{4^x}{\ln4}+C_1$ $\qquad \int 2^{-y}dy=\left|\begin{matrix}-y=t\\-dy=dt\\dy=-dt\end{matrix}\right|=-\int 2^t dt=-\frac{2^t}{\ln2}+C=-\frac{2^{-y}}{\ln2}+C$

$-\frac{2^{-y}}{\ln2}=\frac{4^x}{2\ln2}+C_1\quad /\cdot\ln2\cdot2$

$-2\cdot2^{-y}=4^x+C$ opšte rješenje dif. jednačine

$2^{-y}=\frac{4^x+C}{-2}$ ili $4^x=C-2\cdot2^{-y}$

$-y=\log_2\frac{4^x+C}{-2}$ ili $x=\log_4(C-2^{1-y})$ opšte rješenje

$y=\log_2\frac{-2}{4^x+C}$ opšte rješenje

Homogene jednačine prvog reda

Jednačine prvog reda $y'=f(x,y)$ nazivamo homogene, ako $f(x,y)$ možemo napisati kao f-ju jednih vezanih promjenjivih $f(x,y)=\varphi(\frac{y}{x})$, t.j. jednačinu oblika $y'=\varphi(\frac{y}{x})$.

Homogena jednačina se svodi na jednačinu sa razdvojenim promjenjivim koju možemo riješiti zamjenom x-a (ili y-na) novom f-jom u po formuli $y=ux$ (ili $x=uy$).

Homogene diferencijalne jednačine

su oblika $y'=f(\frac{y}{x})$, uvodimo smjenu $\frac{y}{x}=u \Rightarrow y=ux,\ y'=u'x+u$

1. Riješiti diferencijalnu jednačinu $xy'+y=-x$.

Rj. $xy'+y=-x \quad /:x\ (x\neq 0)$

$y'+\frac{y}{x}=-1$

$y'=-1-\frac{y}{x}$ ovo je hom. dif. jedn.

uvodimo smjenu $u=\frac{y}{x}$

$y=ux,\ y'=u'x+u$

$u'x+u=-1-u$

$\frac{du}{dx}x=-1-2u \quad /\cdot\frac{dx}{(-1-2u)\cdot x}$

$\frac{du}{-1-2u}=\frac{dx}{x}$

$\frac{du}{2u+1}=-\frac{dx}{x} \quad \int\int$ $\quad \left|\begin{array}{l} 2u=t \\ 2du=dt \\ du=\frac{1}{2}dt \end{array}\right|$

$\frac{1}{2}\ln|2u+1|=-\ln|x|+\ln|c_1| \quad /\cdot 2$

$\ln|2u+1|=2\ln|\frac{c_1}{x}|$

$2u+1=\left(\frac{c_1}{x}\right)^2$

$2u=\frac{c}{x^2}-1 \Rightarrow 2\frac{y}{x}=\frac{c}{x^2}-1 \Rightarrow 2y=\frac{c}{x^2}-x$ t.j. $y=\frac{c}{x^2}-\frac{x}{2}$ opšte rješenje diferenc. jedn.

2. Naći partikularno rješenje diferencijalne jednačine $xy'=y(1+\ln y-\ln x)$ tako da zadovoljava uslov $y(1)=e$.

Rj. $xy'=y(1+\ln\frac{y}{x}) \quad /:x$

$y'=\frac{y}{x}(1+\ln\frac{y}{x})$ ovo je hom. dif. jedn.

$u=\frac{y}{x} \Rightarrow y=ux,\ y'=u'x+u$

$u'x+u=u(1+\ln u)$

$u'x=u\ln u,\quad u'=\frac{du}{dx}$

$\frac{du}{u\ln u}=\frac{dx}{x} \quad \int\int$

$\int\frac{du}{u\ln u}=\left|\begin{array}{l}\ln u=t\\ \frac{du}{u}=dt\end{array}\right|=\int\frac{dt}{t}=\ln|t|=\ln\ln u$

$\ln\ln u=\ln x+\ln c$

$\ln u=xc \Rightarrow u=e^{cx}$

$y=x\,e^{cx}$ opšte rješenje dif. jedn.

$\left.\begin{array}{l}y(1)=e\\ y(1)=1\cdot e^{c\cdot 1}\end{array}\right\} \Rightarrow e^c=e \Rightarrow c=1$

$y=xe^x$ partikularno rješenje dif. jedn.

3. Naći opšte rješenje dif. jednačine $xy'=xe^{\frac{y}{x}}+y$.

Rj. $y=-x\ln\ln\frac{c}{x}$

④ Riješiti diferencijalnu jednačinu

$$y^3y' + 3xy^2 + 2x^3 = 0.$$

Rj.

$$y^3y' + 3xy^2 + 2x^3 = 0$$

$$y^3y' = -3xy^2 - 2x^3 \quad / : y^3$$

$$y' = \frac{-3xy^2 - 2x^3}{y^3} \quad \begin{matrix} :x^3 \\ :x^3 \end{matrix}$$

$$y' = \frac{-3\left(\frac{y}{x}\right)^2 - 2}{\left(\frac{y}{x}\right)^3}$$

ovo je homogena diferencijalna jednačina

uvodimo smjenu $\frac{y}{x} = u$

tj. $y = ux$

$y' = u'x + u$

$$u'x + u = \frac{-3u^2 - 2}{u^3}$$

$$u'x = \frac{-3u^2 - 2}{u^3} - u$$

$$u'x = \frac{-3u^2 - 2 - u^4}{u^3}$$

$$\frac{du}{dx}x = \frac{-u^4 - 3u^2 - 2}{u^3}$$

$$\frac{u^3}{-u^4 - 3u^2 - 2}\,du = \frac{dx}{x}$$

$$\frac{u^3}{u^4 + 3u^2 + 2}\,du = -\frac{dx}{x}$$

$u^4 + 3u^2 + 2 = 0$

$u^2 = t, \quad t^2 + 3t + 2 = 0 \qquad t_{1,2} = \frac{-3 \pm 1}{2}$

$D = 9 - 8 = 1 \qquad t_1 = \frac{-4}{2} = -2$

$(u^2+2)(u^2+1) = 0 \qquad t_2 = \frac{-2}{2} = -1$

$$\frac{u^3}{u^4 + 3u^2 + 2} = \frac{Au + B}{u^2 + 2} + \frac{Cu + D}{u^2 + 1} \quad /(u^2+2)(u^2+1)$$

$$u^3 = A(u^3 + u) + B(u^2 + 1) + C(u^3 + 2u) + D(u^2 + 2)$$

$$\begin{aligned} A + C &= 1 \\ B + D &= 0 \\ A + 2C &= 0 \\ B + 2D &= 0 \\ \hline B = D &= 0 \end{aligned}$$

$$\begin{aligned} A + C &= 1 \\ A + 2C &= 0 \quad / \cdot (-1) \\ \hline A + C &= 1 \\ -A - 2C &= 0 \\ \hline -C &= 1 \\ C &= -1 \end{aligned} \qquad \therefore A = 2$$

$$\frac{u^3}{u^4 + 3u^2 + 2} = \frac{2u}{u^2 + 2} + \frac{-u}{u^2 + 1}$$

$$\frac{u^3}{u^4 + 3u^2 + 2}\,du = -\frac{dx}{x} \quad / \int$$

$$\ln|u^2 + 2| - \frac{1}{2}\ln|u^2 + 1| = -\ln|x| + \ln c$$

$$\ln\frac{|u^2 + 2|}{\sqrt{u^2 + 1}} = \ln\frac{c}{x}$$

$$\frac{u^2 + 2}{\sqrt{u^2 + 1}} = \frac{c}{x}$$

$$\frac{\left(\frac{y}{x}\right)^2 + 2}{\sqrt{\left(\frac{y}{x}\right)^2 + 1}} = \frac{c}{x}$$

rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu

$$(3y^2+3xy+x^2)\,dx=(x^2+2xy)\,dy$$

Rj:

$$(x^2+2xy)\,dy=(3y^2+3xy+x^2)\,dx \quad /:dx \quad /:(x^2+2xy)$$

$$\frac{dy}{dx}=\frac{3y^2+3xy+x^2}{x^2+2xy}\quad \begin{matrix}:x^2\\ :x^2\end{matrix}$$

$$y'=\frac{3\left(\frac{y}{x}\right)^2+3\frac{y}{x}+1}{2\frac{y}{x}+1}$$

ovo je homogena difer. jedn. uvodimo smjenu $u=\frac{y}{x}$

$$y=ux \quad /\frac{d}{dx}$$

$$y'=u'x+u$$

$$u'x+u=\frac{3u^2+3u+1}{2u+1}$$

$$u'x=\frac{3u^2+3u+1}{2u+1}-u$$

$$u'x=\frac{3u^2+3u+1-2u^2-u}{2u+1}$$

$$u'x=\frac{u^2+2u+1}{2u+1}$$

$$\frac{du}{dx}x=\frac{u^2+2u+1}{2u+1}$$

$$\frac{2u+1}{u^2+2u+1}\,du=\frac{dx}{x} \quad \dots(*)$$

$$\int\frac{2u+1}{u^2+2u+1}\,du=\int\frac{2u+2-1}{u^2+2u+1}\,du=\int\frac{2u+2}{u^2+2u+1}\,du-\int\frac{du}{u^2+2u+1}=$$

$$=\left|\begin{matrix}u^2+2u+1=t\\(2u+2)\,du=dt\end{matrix}\right|=\int\frac{dt}{t}-\int\frac{du}{(u+1)^2}=\left|\begin{matrix}u+1=s\\du=ds\end{matrix}\right|=$$

$$\ln|t|-\int\frac{ds}{s^2}=\ln|u^2+2u+1|-\frac{s^{-1}}{(-1)}+c=\ln(u+1)^2+\frac{1}{u+1}+c$$

$$(*)\Rightarrow \ln(u+1)^2+\frac{1}{u+1}=\ln|x|+c$$

$$\ln\left(\frac{y}{x}+1\right)^2+\frac{1}{\frac{y}{x}+1}=\ln|x|+c$$ rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu

$$(5y+7x)\,dy+(8y+10x)\,dx=0$$

Rj:

$$(5y+7x)\,dy+(8y+10x)\,dx=0$$

$$(5y+7x)\,dy=(-8y-10x)\,dx=0$$

$$\frac{dy}{dx}=\frac{-8y-10x}{5y+7x}\quad \begin{matrix}/:x\\ /:x\end{matrix}$$

$$y'=\frac{-8\left(\frac{y}{x}\right)-10}{5\left(\frac{y}{x}\right)+7}$$

ovo je homogena diferencijalna jednačina, uvodimo smjenu $u=\frac{y}{x}$

$$y=u\cdot x \quad /\frac{d}{dx}$$

$$y'=u'x+u$$

$$u'x+u=\frac{-8u-10}{5u+7}$$

$$u'x=\frac{-8u-10}{5u+7}-u$$

$$u'x=\frac{-8u-10-u(5u+7)}{5u+7}$$

$$u'x=\frac{-8u-10-5u^2-7u}{5u+7}$$

$$\frac{du}{dx}x=\frac{\overbrace{-5u^2-15u-10}^{(-5)(u^2+3u+2)}}{5u+7}$$

$$\frac{du}{dx}x=(-5)\frac{(u+1)(u+2)}{5u+7}$$

$$\frac{(5u+7)\,du}{u^2+3u+2}=-5\frac{dx}{x} \quad \dots(*)$$

$$\frac{5u+7}{u^2+3u+2}=\frac{5u+7}{(u+1)(u+2)}=\frac{A}{u+1}+\frac{B}{u+2} \quad /(u+1)(u+2)$$

$$5u+7=A(u+2)+B(u+1)$$

$$\begin{aligned}A+B&=5\\ -\ 2A+B&=7\end{aligned} \qquad \begin{aligned}-A&=-2\\ A&=2\end{aligned} \qquad B=3$$

$$\int\frac{5u+7}{u^2+3u+2}\,du=2\int\frac{du}{u+1}+3\int\frac{du}{u+2}$$

$$(*)\Rightarrow 2\ln|u+1|+3\ln|u+2|=-5\ln|x|+\ln|C|$$

$$\ln(u+1)^2(u+2)^3=\ln(x^{-5}C)$$

$$(u+1)^2(u+2)^3=\frac{C}{x^5}$$

$$\left(\frac{y}{x}+1\right)^2\left(\frac{y}{x}+2\right)^3=\frac{C}{x^5}$$ rješenje diferencijalne jednačine

Diferencijalne jednačine koje se svode na homogene

su oblika $y' = f\left(\dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$

ako je $a_1b_2 - a_2b_1 = 0$ uvodimo smjenu $a_1x + b_1y = u$ i dobijamo dif. jedn. sa razdvojenim promjenjivim.

ako je $a_1b_2 - a_2b_1 \neq 0$ uvodimo smjenu $x = u + \alpha$, $y = v + \beta$ gdje α i β dobijamo iz sistema
$$a_1\alpha + b_1\beta + c_1 = 0$$
$$a_2\alpha + b_2\beta + c_2 = 0 .$$

1. Riješiti diferencijalnu jednačinu $(x-2y+1)y' = 2x - y + 1$.

Rj. $y' = \dfrac{2x-y+1}{x-2y+1}$, $a_1b_2 - a_2b_1 \neq 0 \Rightarrow x = u+\alpha,\ y = v+\beta \Rightarrow y' = v'$

$$\left.\begin{array}{l} 2\alpha - \beta + 1 = 0 \\ \alpha - 2\beta + 1 = 0 \end{array}\right\} \Rightarrow \alpha = -\frac{1}{3} \ \text{ i } \ \beta = \frac{1}{3} \qquad x = u - \frac{1}{3},\quad y = v + \frac{1}{3}$$

$$v' = \frac{2(u-\frac{1}{3}) - (v+\frac{1}{3}) + 1}{(u-\frac{1}{3}) - 2(v+\frac{1}{3}) + 1}$$

$$v' = \frac{2u - v}{u - 2v} \quad \begin{array}{l} /:u \\ /:u \end{array}$$

$$v' = \frac{2 - \frac{v}{u}}{1 - 2\frac{v}{u}}$$ ovo je hom. dif. jedn.

smjena $\frac{v}{u} = z$, $v = uz$, $v' = z'u + z$

$$z'u + z = \frac{2-z}{1-2z}$$

$$z'u = \frac{2(z^2 - z + 1)}{1-2z}, \quad z' = \frac{dz}{du}$$

$$\frac{1-2z}{z^2-z+1}\,dz = 2\frac{du}{u} \quad \Big/ \int$$

$$-\ln(z^2 - z + 1) = 2\ln u + \ln C_1$$

$$\ln\frac{1}{z^2-z+1} = \ln C_1 u^2$$

$$1 = C_1 u^2 (z^2 - z + 1)$$

$$1 = C_1 u^2 \left(\frac{v^2}{u^2} - \frac{v}{u} + 1\right) \quad /:C_1$$

$$C = v^2 - uv + u^2$$

$$C = \left(y - \frac{1}{3}\right)^2 - \left(x + \frac{1}{3}\right)\left(y - \frac{1}{3}\right) + \left(x + \frac{1}{3}\right)^2$$ opšte rješenje diferenc. jednač.

2.^{v} Riješiti diferencijalnu jednačinu $(2x+y+1)y' = 4x + 2y + 3$.

Rj. $\ln Cx^{16}(8x + 4y + 5) = 4(2x + y + 1)$ opšte rješenje

3.^{v} Riješiti diferencijalnu jednačinu $(2x - 4y + 6)dx + (x + y - 3)dy = 0$

Rj. $(y - 2x)^3 = C(y - x - 1)^2$ opšte rješenje

(#) Riješiti diferencijalnu jednačinu
$$(x - y - 2)dx + (2x - y - 5)dy = 0$$

Rj.
$$(2x - y - 5)\,dy = -(x - y - 2)\,dx \quad \Big/ \cdot \frac{1}{dx} \cdot \frac{1}{2x-y-5}$$

$$\frac{dy}{dx} = \frac{-x + y + 2}{2x - y - 5}$$

$$y' = \frac{-x + y + 2}{2x - y - 5}$$ diferencijalna jednačina koja se svodi na homogenu $y' = f\left(\dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$

$a_1b_2 - a_2b_1 = 1 - 2 = -1 \neq 0$

uvodimo smjenu $x = u + \alpha$, $y = v + \beta$

$$\begin{array}{r} -\alpha + \beta + 2 = 0 \\ +\ 2\alpha - \beta - 5 = 0 \\ \hline \alpha - 3 = 0 \\ \alpha = 3 \end{array} \qquad \begin{array}{l} -\alpha + \beta + 2 = 0 \\ -3 + \beta + 2 = 0 \\ \beta = 1 \end{array} \qquad \begin{array}{l} x = u + 3 \\ y = v + 1 \\ y' = v' \end{array} \Rightarrow \begin{array}{l} u = x - 3 \\ v = y - 1 \end{array}$$

$$v' = \frac{-u - 3 + v + 1 + 2}{2u + 6 - v - 1 - 5}$$

$$v' = \frac{-u + v}{2u - v} \quad \begin{array}{l} /:u \\ /:u \end{array}$$

$$v' = \frac{-1 + \frac{v}{u}}{2 - \frac{v}{u}}$$ ovo je homogena diferenc. jednačina

uvodimo smjenu $z = \frac{v}{u}$

$v = z \cdot u \quad /'_u$

$v' = z' \cdot u + z$

$$z' \cdot u + z = \frac{-1 + z}{2 - z}$$

$$z'u = \frac{-1+z}{2-z} - z$$

$$z'u = \frac{-1 + z - 2z + z^2}{2 - z}$$

$$z'u = \frac{z^2 - z - 1}{2 - z}, \quad z' = \frac{dz}{du}$$

$$\frac{2-z}{z^2-z-1}\,dz = \frac{du}{u} \quad \Big/ \int$$

$$2 - z = (-1)(z-2) = \left(-\frac{1}{2}\right)(2z - 4) = \left(-\frac{1}{2}\right)(2z - 1 - 3)$$

$$\int \frac{2-z}{z^2-z-1}\,dz = -\frac{1}{2}\int\frac{2z-1}{z^2-z-1}\,dz + \frac{3}{2}\int\frac{dz}{z^2-z-1} \overset{(*)}{=} -\frac{1}{2}\ln(z^2-z-1) + \frac{3\sqrt{5}}{10}\ln\left|\frac{2z-1-\sqrt{5}}{2z-1+\sqrt{5}}\right| + C_1$$

$$\frac{3}{2}\int\frac{dz}{z^2-z-1} = \left| \begin{array}{l} z^2 - z - 1 = z^2 - 2\cdot\frac{1}{2}\cdot z + \frac{1}{4} - \frac{1}{4} - 1 = \\ = \left(z - \frac{1}{2}\right)^2 - \frac{5}{4} \end{array} \right|$$

$$= \frac{3}{2}\int\frac{dz}{(z-\frac{1}{2})^2 - \frac{5}{4}} = \left| \begin{array}{l} z - \frac{1}{2} = \frac{\sqrt{5}}{2}t \\ dz = \frac{\sqrt{5}}{2}dt \\ t = \frac{2z-1}{\sqrt{5}} \end{array} \right| = \frac{3}{2}\cdot\frac{\sqrt{5}}{2}\cdot\frac{4}{5}\int\frac{dt}{t^2 - 1}$$

$$= \frac{3\sqrt{5}}{5}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{3\sqrt{5}}{10}\ln\left|\frac{2z-1-\sqrt{5}}{2z-1+\sqrt{5}}\right| \quad \dots(*)$$

$$\ln u = -\frac{1}{2}\ln(z^2 - z - 1) + \frac{3\sqrt{5}}{10}\ln\left|\frac{2z-1-\sqrt{5}}{2z-1+\sqrt{5}}\right| + \ln C_2 \quad /\cdot 10$$

$$u^{10} = \frac{C}{(z^2 - z - 1)^5}\left|\frac{2z-1-\sqrt{5}}{2z-1+\sqrt{5}}\right|^{3\sqrt{5}}$$ opšte rješenje difer. jedn.

$z = \frac{v}{u}$, $v = y - 1$, $u = x - 3$

Linearne diferencijalne jednačine i jednačina Bernulija

Jednačina oblika $y' + P(x)\,y = Q(x)$, gdje su $P(x)$ i $Q(x)$ date funkcije po x, koja je linearna (prvog stepena) u odnosu na f-ju y i njezin izvod y', nazivamo linearna diferencijalna jednačina.

Pomoću zamjene f-je y sa proizvodom dvije pomoćne f-je $y=uv$, linearna jednačina se svodi na dvije jednačine sa razdvojenim promjenjivim u odnosu na svaku od pomoćnih f-ja.

Jednačina Bernulija $y' + P(x)\,y = y^n Q(x)$, koja se razlikuje od linearne jednačine u činjenici da je desna strana faktor nekog stepena f-je y, rješava se na potpuno isti način kao i linearna. Pomoću smjene $y=uv$ ona se također svodi na dvije jednačine sa razdvojenim promjenjivim.

Linearna diferencijalna jednačina

su oblika $y' + p(x)\cdot y = g(x)$. uvodimo smjenu $y = uv$.

1. Riješiti diferencijalnu jednačinu $(1+x^2)\,y' = x(2y+1)$.

Rj. $(1+x^2)\,y' = 2xy + x$

$(1+x^2)\,y' - 2xy = x \quad /:(1+x^2)$

$$y' - \frac{2x}{1+x^2}\,y = \frac{x}{1+x^2}$$ ovo je lin. dif. jedn.

$y = u\cdot v$, $y' = u'\cdot v + u\cdot v'$

uvrstimo smjenu

$$u'\cdot v + u\cdot v' - \frac{2x}{1+x^2}\,uv = \frac{x}{1+x^2}$$

$$u'\cdot v + \underbrace{u\cdot\left(v' - \frac{2x}{1+x^2}\,v\right)}_{\text{ovaj dio izjednačimo sa 0 da bi našli } v} = \frac{x}{1+x^2}$$

a)
$$v' - \frac{2x}{1+x^2}\,v = 0, \quad v' = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{2x}{1+x^2}\,v$$

$$\frac{dv}{v} = \frac{2x}{1+x^2}\,dx \quad \Big/ \int$$

$$\int \frac{dv}{v} = 2\int \frac{x}{1+x^2}\,dx \quad \left|\begin{array}{l} 1+x^2 = t \\ 2x\,dx = dt \end{array}\right|$$

$$\ln|v| = \ln|1+x^2|$$

$$v = 1+x^2$$

b)
$$u'v = \frac{x}{1+x^2}$$

$$u'(1+x^2) = \frac{x}{1+x^2}, \quad u' = \frac{du}{dx}$$

$$\frac{du}{dx} = \frac{x}{(1+x^2)^2}, \quad du = \frac{x}{(1+x^2)^2}\,dx \quad \Big/ \int$$

$$\int du = \int \frac{x}{(1+x^2)^2}\,dx \quad \left|\begin{array}{l} 1+x^2 = t \\ 2x\,dx = dt \\ x\,dx = \frac{1}{2}dt \end{array}\right|$$

$$u = -\frac{1}{2(1+x^2)} + C$$

$$y = u\cdot v = \left[-\frac{1}{2(1+x^2)} + C\right](1+x^2)$$

$$y = C(1+x^2) - \frac{1}{2}$$ opšte rješenje diferencijalne jednačine

2. Riješiti diferencijalnu jednačinu $xy' - \dfrac{y}{x+1} = x$ ako je $y(1) = -1$.

Rj. $y = \dfrac{x}{x+1}(x + \ln|x| + C)$ opšte rješenje diferenc. jedn.

$y = \dfrac{x}{x+1}(x + \ln|x| - 3)$ partikularno rješenje diferenc. jedn.

3. Riješiti diferencijalnu jednačinu

$y' + y\cos x = 0{,}5\sin 2x$

Rj. $y = 1 - \sin x + Ce^{-\sin x}$ opšte rješenje dif. jedn.

Riješiti diferencijalnu jednačinu

$$y' - \frac{xy}{1+x^2} = \frac{x\sqrt{1+x^2}}{x^2-2x+2}.$$

Rj. $y' - \frac{x}{1+x^2}y = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$ ovo je linearna diferencijalna jednačina uvodimo smjenu $y=uv$

$y=uv,\ y'=u'v+uv'$

$$u'v+uv' - \frac{x}{1+x^2}uv = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

$$u'v+u\underbrace{\left(v'-\frac{x}{1+x^2}v\right)}_{=0} = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

a)

$$v' - \frac{x}{1+x^2}v = 0$$

$$\frac{dv}{dx} = \frac{x}{1+x^2}v \quad /:v$$

$$\frac{dv}{v} = \frac{x}{1+x^2}dx \quad /\int$$

$$\int\frac{dv}{v} = \int\frac{x}{1+x^2}dx$$

$$\int\frac{x}{1+x^2}dx = \left|\begin{matrix}1+x^2=t\\ 2x\,dx=dt\\ x\,dx=\frac{1}{2}dt\end{matrix}\right| = \frac{1}{2}\int\frac{dt}{t}$$

$$= \frac{1}{2}\ln|t|+c = \ln|1+x^2|^{\frac{1}{2}}+c$$

$$\ln|v| = \ln\sqrt{1+x^2}$$

$$v = \sqrt{1+x^2}$$

b)

$$u'v = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

$$\frac{du}{dx}\sqrt{1+x^2} = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

$$du = \frac{x}{x^2-2x+2}dx \quad /\int$$

$$\int\frac{x\overset{-1+1}{}}{x^2-2x+2}dx = \int\frac{x-1}{x^2-2x+2}dx + \int\frac{dx}{x^2-2x+2}$$

$$= \left|\begin{matrix}x^2-2x+2=t\\ (2x-2)dx=dt\\ (x-1)dx=\frac{1}{2}dt\end{matrix}\quad \begin{matrix}x^2-2x+2=\\ x^2-2x+1+1=\\ =(x-1)^2+1\end{matrix}\right|$$

$$= \frac{1}{2}\int\frac{dt}{t} + \int\frac{dx}{(x-1)^2+1} =$$

$$= \frac{1}{2}\ln|t| + \operatorname{arctg}(x-1)+c$$

$$= \frac{1}{2}\ln|x^2-2x+2| + \operatorname{arctg}(x-1)+c$$

$$u = \frac{1}{2}\ln|x^2-2x+2| + \operatorname{arctg}(x-1)+c$$

$$y=uv = \left(\frac{1}{2}\ln|x^2-2x+2|+\operatorname{arctg}(x-1)+c\right)\sqrt{1+x^2} =$$

$$= c\sqrt{1+x^2} + \sqrt{1+x^2}\left(\frac{1}{2}\ln|x^2-2x+2|+\operatorname{arctg}(x-1)\right)$$

rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu $(x^2+2x-2y)dx - dy = 0$.

Rj. $(x^2+2x-2y)dx - dy = 0 \quad /:dx$

$$x^2+2x-2y-y' = 0$$

$$y'+2y = x^2+2x$$ Ovo je linearna diferencijalna jednačina

Uvodimo smjenu $y=uv \quad /\frac{d}{dx}$

$$y'=u'v+uv'$$

$$u'v+uv'+2uv = x^2+2x$$

$$u'v+u\underbrace{(v'+2v)}_{=0} = x^2+2x$$

a) $v'+2v=0$

$$\frac{dv}{dx} = -2v$$

$$\frac{dv}{v} = -2dx \quad /\int$$

$$\ln v = -2x$$

$$v = e^{-2x}$$

b) $u'v+u\cdot 0 = x^2+2x$

$$u'v = x^2+2x$$

$$u'e^{-2x} = x^2+2x$$

$$\frac{du}{dx} = \frac{x^2+2x}{e^{-2x}}$$

$$du = \frac{x^2+2x}{e^{-2x}}dx$$

$$du = (x^2+2x)e^{2x}dx \quad \ldots(*)$$

$2x=t$, $2dx=dt$, $dx=\frac{1}{2}dt$

$$\int(x^2+2x)e^{2x}dx = \left|\begin{matrix}u=x^2+2x & dv=e^{2x}dx\\ du=2x+2 & v=\frac{1}{2}e^{2x}\end{matrix}\right| = \frac{1}{2}e^{2x}(x^2+2x) - \int(x+1)e^{2x}dx$$

$$\int(x+1)e^{2x}dx = \left|\begin{matrix}u=x+1 & dv=e^{2x}dx\\ du=dx & v=\frac{1}{2}e^{2x}\end{matrix}\right| = \frac{1}{2}(x+1)e^{2x} - \frac{1}{2}\int e^{2x}dx$$

$$\int(x^2+2x)e^{2x}dx = \frac{1}{2}e^{2x}(x^2+2x) - \frac{1}{2}e^{2x}(x+1) + \frac{1}{4}e^{2x}+c$$

$$= \frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}+c$$

$$(*) \Rightarrow \quad u = \frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}+c$$

$$y=uv = \left(\frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}+c\right)e^{-2x} =$$

$$= \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4} + ce^{-2x}$$ opšte rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $y'\cos x - y\sin x = x^3 e^{x^2}$ uz početni uslov $y(0)=1$.

Rj. $y'\cos x - y\sin x = x^3 e^{x^2} \quad /:\cos x$

$y' - y\,tg x = \frac{x^3 e^{x^2}}{\cos x}$ ovo je linearna diferencijalna jednačina

uvodimo smjenu

$y = uv$

$y' = u'v + uv'$

$u'v + uv' - uv\,tg x = \frac{x^3 e^{x^2}}{\cos x}$

$u'v + \underbrace{u(v' - v\,tg x)}_{=0} = \frac{x^3 e^{x^2}}{\cos x}$

$v' - v\,tg x = 0$

$\frac{dv}{dx} = v\,tg x$

$\frac{dv}{v} = tg x\,dx$

$\int \frac{dv}{v} = \int tg x\,dx$

$\ln v = \ln\left|\frac{1}{\cos x}\right|$

$v = \frac{1}{\cos x}$

$$\int tg x\,dx = \left| \begin{matrix} tg x = t \\ x = arctg\,t \\ dx = \frac{dt}{1+t^2} \end{matrix} \right| = \int \frac{t}{1+t^2}dt = \left| \begin{matrix} 1+t^2 = s \\ 2t\,dt = ds \\ t\,dt = \frac{ds}{2} \end{matrix} \right| = \frac{1}{2}\int \frac{ds}{s} =$$

$$= \frac{1}{2}\ln|s| = \frac{1}{2}\ln|1+t^2| =$$

$$= \frac{1}{2}\ln|1+tg^2 x| = \frac{1}{2}\ln\left|1 + \frac{\sin^2 x}{\cos^2 x}\right|$$

$$= \frac{1}{2}\ln\left|\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right| = \ln\left|\frac{1}{\cos x}\right|$$

$u'v = \frac{x^3 e^{x^2}}{\cos x}$

$u' \cdot \frac{1}{\cos x} = \frac{x^3 e^{x^2}}{\cos x} \quad /\cdot\cos x$

$\frac{du}{dx} = x^3 e^{x^2}$

$du = x^3 e^{x^2}dx$

$$I = \int x^3 e^{x^2}dx = \left| \begin{matrix} u = x^2 & dv = xe^{x^2}dx \\ du = 2x & v = \int xe^{x^2}dx = \left|\begin{matrix} x^2 = t \\ 2x\,dx = dt \\ x\,dx = \frac{dt}{2}\end{matrix}\right| = \frac{1}{2}\int e^t dt = \frac{1}{2}e^{x^2} \end{matrix} \right| =$$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}\cdot 2\int x\,e^{x^2} = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$$

$du = x^3 e^{x^2}dx \quad /\int$

$u = \frac{1}{2}e^{x^2}(x^2-1) + c_1$

$y = uv = \frac{e^{x^2}(x^2-1)+c}{2\cos x}$

opće rješenje diferencijalne jednačine

$y(0) = 1$

$y(0) = \frac{e^0(0-1)+c}{2\cos 0} = \frac{-1+c}{2} = 1$

$-1 + c = 2$

$c = 3$

$y = \frac{e^{x^2}(x^2-1)+3}{2\cos x}$

partikularno rješenje diferencijalne jednačine

Bernulijeva diferencijalna jednačina

su oblika $y' + p(x)y = q(x)y^n$, $n \in R$, $n \neq 0$ i $n \neq 1$

ove dif. jedn. rješavamo na isti način kao što smo rješavali linearne dif. jedn.

uvodimo smjenu $y = uv$

(1.) Riješiti diferencijalnu jednačinu $xy' - x^2\sqrt{y} = 4y$.

Rj. $xy' - 4y = x^2\sqrt{y} \quad /:x$

$y' - \frac{4}{x}y = x\sqrt{y}$ ovo je Bern. dif. jedn.

smjena $y = uv$

$y' = u'v + uv'$

$u'v + uv' - \frac{4}{x}uv = x\sqrt{uv}$

$u'v + \underbrace{u(v' - v\frac{4}{x})}_{=0} = x\sqrt{uv}$

(da bi smo našli v)

a) $v' - v\frac{4}{x} = 0 \Rightarrow v' = v\frac{4}{x}$

$v' = \frac{dv}{dx}, \quad \frac{dv}{v} = \frac{4}{x}dx \quad //\int$

$\int\frac{dv}{v} = \int\frac{4}{x}dx \Rightarrow \ln|v| = 4\ln|x|$

$v = x^4$

b) $u'v = x\sqrt{uv}$

$u'x^4 = x\sqrt{ux^4}, \quad x^4u' = x^3\sqrt{u}$

$\frac{du}{dx} = x^{-1}\sqrt{u}, \quad \frac{du}{\sqrt{u}} = \frac{dx}{x} \quad //\int$

$\int\frac{du}{\sqrt{u}} = \int\frac{dx}{x} \Rightarrow 2\sqrt{u} = \ln x + c$

$\sqrt{u} = \frac{\ln x + c}{2}$

$u = \frac{(\ln x + c)^2}{4}$

$y = uv = \frac{(\ln x + c)^2}{4}\cdot x^4$

$y = \frac{x^4}{4}(\ln x + c)^2$ opšte rješenje difer. jedn.

(2.) Naći partikularno rješenje diferencijalne jednačine $y' = xy^3 - y$ koje prolazi kroz tačku $A(0,1)$.

Rj: $y^{-2} = e^{2x}\left[e^{-2x}(x+\frac{1}{2}) + c\right]$ opšte rješenje dif. jedn.

$y^{-2} = \frac{1}{2}e^{2x} + x + \frac{1}{2}$ partikul. rješ. dif. jedn.

(3.) Riješiti diferencijalnu jednačinu

$(1-x^2)y' = xy + xy^2$

Rj. $y = \frac{c}{\sqrt{1-x^2}} - 1$

Riješiti diferencijalnu jednačinu

$$y' + \frac{y}{4x} + y^3 e^{\sqrt{x}} = 0 \quad \text{ako je} \quad y(1) = 1.$$

Rj. $y' + \frac{1}{4x} y = -e^{\sqrt{x}} y^3$ ovo je Bernulijeva diferencijalna jednačina.

uvodimo smjenu $y = uv$

$y' = u'v + uv'$

$$u'v + uv' + \frac{1}{4x} uv = -e^{\sqrt{x}} u^3 v^3$$

$$u'v + u\underbrace{\left(v' + \frac{1}{4x} v\right)}_{=0} = -e^{\sqrt{x}} u^3 v^3$$

a) $v' + \frac{1}{4x} v = 0$

$$\frac{dv}{dx} = \frac{-v}{4x}$$

$$\frac{dv}{v} = \frac{-dx}{4x}$$

$$\frac{dv}{v} = -\frac{1}{4} \cdot \frac{dx}{x} \quad \int$$

$$\ln v = -\frac{1}{4} \ln|x|$$

$$\ln v = \ln|x|^{-\frac{1}{4}}$$

$$v = \frac{1}{\sqrt[4]{x}}$$

b) $u'v = -e^{\sqrt{x}} u^3 v^3$

$$u' \cdot \frac{1}{\sqrt[4]{x}} = -e^{\sqrt{x}} u^3 \frac{1}{\sqrt[4]{x^3}} \quad / \cdot \sqrt[4]{x}$$

$$\frac{du}{dx} = -e^{\sqrt{x}} \frac{u^3}{\sqrt[4]{x^2}}$$

$$\frac{du}{u^3} = -\frac{e^{\sqrt{x}}}{\sqrt[4]{x^2}} dx$$

$$\frac{du}{u^3} = -\frac{e^{\sqrt{x}}}{\sqrt{x}} dx \quad \int$$

$$\left(e^{\sqrt{x}}\right)' = e^{\sqrt{x}} \cdot (\sqrt{x})' = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$\frac{u^{-2}}{-2} = -2e^{\sqrt{x}} + C_1 \quad / \cdot (-2)$$

$$\frac{1}{u^2} = 4e^{\sqrt{x}} + C.$$

$$u^2 = \frac{1}{4e^{\sqrt{x}} + C} \Rightarrow u = \frac{1}{\sqrt{4e^{\sqrt{x}} + C}}$$

$$y = uv = \frac{1}{\sqrt[4]{x}\sqrt{4e^{\sqrt{x}} + C}}$$ opšte rješenje diferenc. jedn.

$$y(1) = 1 \Rightarrow \frac{1}{\sqrt{4e + C}} = 1 \qquad \sqrt{4e + C} = 1$$

$$4e + C = 1 \Rightarrow C = 1 - 4e$$

$$y = \frac{1}{\sqrt[4]{x}\sqrt{4e^{\sqrt{x}} + 1 - 4e}}$$ partikularno rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu $y' = y^4 \cos x + y \operatorname{tg} x$.

Rj. $y' - y \operatorname{tg} x = \cos x \, y^4$ ovo je Bernulijeva diferencijalna jednačina uvodimo smjenu $y = uv$, $y' = u'v + uv'$.

$$u'v + uv' - uv \operatorname{tg} x = u^4 v^4 \cos x$$

$$u'v + u\underbrace{(v' - v \operatorname{tg} x)}_{=0} = u^4 v^4 \cos x$$

$$v' - v \operatorname{tg} x = 0$$

$$v' = v \operatorname{tg} x$$

$$\frac{dv}{dx} = v \operatorname{tg} x$$

$$\frac{dv}{v} = \operatorname{tg} x \, dx \quad \int$$

$$\ln v = \ln \frac{1}{\cos x}$$

$$v = \frac{1}{\cos x}$$

$$\int \operatorname{tg} x \, dx = \left| \begin{matrix} \operatorname{tg} x = t \\ x = \operatorname{arctg} t \\ dx = \frac{dt}{1+t^2} \end{matrix} \right| = \int \frac{t}{1+t^2} dt = \left| \begin{matrix} 1+t^2 = s \\ 2t\,dt = ds \\ t\,dt = \frac{ds}{2} \end{matrix} \right| = \frac{1}{2}\int \frac{ds}{s} = \frac{1}{2}\ln|s| + C = \frac{1}{2}\ln|1+t^2| + C$$

$$= \frac{1}{2}\ln\left|1 + \frac{\sin^2 x}{\cos^2 x}\right| + C = \frac{1}{2}\ln\left|\frac{1}{\cos^2 x}\right| + C = \ln\frac{1}{\cos x} + C$$

$$u'v = u^4 v^4 \cos x \quad /:v$$

$$u' = u^4 v^3 \cos x$$

$$u' = u^4 \cdot \frac{1}{\cos^2 x}$$

$$\frac{u'}{u^4} = \frac{1}{\cos^2 x}, \quad u' = \frac{du}{dx}$$

$$\frac{du}{u^4} = \frac{dx}{\cos^2 x} \quad \int$$

$$\int \frac{du}{u^4} = \int \frac{dx}{\cos^2 x}$$

$$\int u^{-4} du = \int \frac{dx}{\cos^2 x}$$

$$\frac{u^{-3}}{-3} = \operatorname{tg} x + C_1$$

$$\frac{1}{u^3} = -3\operatorname{tg} x + C.$$

$$\frac{1}{v} = \cos x$$

$$\frac{1}{u^3} = -3 \cdot \frac{\sin x}{\cos x} + C$$

$$\frac{1}{v^3} = \cos^3 x$$

$$\frac{1}{u^3} = -3\frac{\sin x}{\cos x} + C$$

$$\frac{1}{y^3} = \frac{1}{u^3 v^3} = -3\frac{\sin x}{\cos x} \cdot \cos^3 x + C \cdot \cos^3 x$$

$$y^{-3} = -3\sin x \cos^2 x + C \cos^3 x$$

rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $y' = \dfrac{3x^2}{x^3+y+1}$.

Rj.

$$y' = \frac{3x^2}{x^3+y+1}$$

$$\frac{dy}{dx} = \frac{3x^2}{x^3+y+1}$$

$$\frac{dx}{dy} = \frac{x^3+y+1}{3x^2}$$

$$x' = \frac{1}{3}x + \frac{1}{3}yx^{-2} + \frac{1}{3}x^{-2}$$

$$x' - \frac{1}{3}x = \left(\frac{1}{3}y + \frac{1}{3}\right)x^{-2}$$

ovo je Bernulijeva diferencijalna jednačina

Bernulijeva diferencijalna jednačina je oblika $y' + p(x)y = q(x)\cdot y^n$, $n \in R$, $n \neq 0$, $n \neq 1$

Uvodimo smjenu $x = uv$, $x' = u'v + uv'$

$$u'v + uv' - \frac{1}{3}uv = \left(\frac{1}{3}y + \frac{1}{3}\right)(uv)^{-2}$$

$$u'v + u\left(v' - \frac{1}{3}v\right) = \left(\frac{1}{3}y + \frac{1}{3}\right)u^{-2}v^{-2}$$

a) $v' - \frac{1}{3}v = 0$

$v' = \frac{1}{3}v$

$$\frac{dv}{dy} = \frac{1}{3}v$$

$$\frac{dv}{v} = \frac{1}{3}dy$$

$$\ln v = \frac{1}{3}y$$

$$v = e^{\frac{1}{3}y}$$

b) $v = e^{\frac{1}{3}y} = e^{\frac{y}{3}}$

$$u' e^{\frac{y}{3}} = \frac{y+1}{3}u^{-2}e^{-\frac{2y}{3}} \quad / e^{-\frac{y}{3}}\cdot u^2$$

$$u^2 u' = \frac{y+1}{3}e^{-y}$$

$$u^2\frac{du}{dy} = \frac{1}{3}ye^{-y} + \frac{1}{3}e^{-y}$$

$$u^2 du = \frac{1}{3}ye^{-y}dy + \frac{1}{3}e^{-y}dy \quad \ldots(1)$$

$$\int e^{-y}dy = \begin{vmatrix} -y = t \\ dy = -dt \end{vmatrix} = \int e^{t}(-dt) = -\int e^{t}dt = -e^{t} + C = -e^{-y} + C$$

Kako je $\int ye^{-y}dy = \begin{vmatrix} u = y & dv = e^{-y}dy \\ du = dy & v = -e^{-y} \end{vmatrix} = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y} + C$

To je kad izvadimo integral od (1):

$$\frac{1}{3}u^3 = -\frac{1}{3}ye^{-y} - \frac{1}{3}e^{-y} + c_1 - \frac{1}{3}e^{-y} \quad /\cdot 3$$

$$u^3 = -ye^{-y} - 2e^{-y} + c$$

$$u = \sqrt[3]{-ye^{-y} - 2e^{-y} + c}$$

$$x = uv$$

$$x = e^{\frac{y}{3}}\sqrt[3]{-ye^{-y} - 2e^{-y} + c}$$

$$x^3 = e^{y}(-ye^{-y} - 2e^{-y} + c)$$

$$x^3 = -y - 2 + ce^{y}$$

opšte rješenje diferencijalne jednačine

Lagranžova diferencijalna jednačina

su oblika $y = x f(y') + g(y')$ uvodimo smjene $y' = p$, $x = uv$

1. Riješiti diferencijalnu jednačinu $y + xy' = 4\sqrt{y'}$.

Rj. $y = x\cdot(-y') + 4\sqrt{y'}$ ovo je Lagr. dif. jedn.

uvodimo smjenu $y' = p$

$$y = -xp + 4\sqrt{p} \quad / \frac{d}{dx}$$

$$y' = -p - xp' + 4\cdot\frac{1}{2\sqrt{p}}\cdot p', \quad y' = p$$

$$2p = p'\left(-x + \frac{2}{\sqrt{p}}\right), \quad p' = \frac{dp}{dx}$$

$$\frac{1}{p'} = \frac{dx}{dp} = x' \qquad \frac{1}{p'} = \frac{-x + \frac{2}{\sqrt{p}}}{2p}$$

$$x' = -\frac{x}{2p} + \frac{1}{p\sqrt{p}}$$

$$x' + \frac{x}{2p} = \frac{1}{p\sqrt{p}}$$ ovo je linear. dif. jedn.

uvodimo smjenu $x = uv$, $x' = u'v + uv'$

$$u'v + uv' + \frac{uv}{2p} = \frac{1}{p\sqrt{p}}$$

$$u'v + u\underbrace{\left(v' + \frac{v}{2p}\right)}_{=0} = \frac{1}{p\sqrt{p}}$$

a) $v' + \frac{v}{2p} = 0$, $\frac{dv}{dp} = -\frac{v}{2p}$

$$\frac{dv}{v} = -\frac{1}{2}\frac{dp}{p} \quad //\int$$

$$\ln|v| = -\frac{1}{2}\ln|p|$$

$$v = p^{-\frac{1}{2}} = \frac{1}{\sqrt{p}}$$

b) $u'\cdot\frac{1}{\sqrt{p}} = \frac{1}{p\sqrt{p}} \quad /\sqrt{p} \Rightarrow u' = \frac{1}{p}$

$$u = \ln|p| + c$$

$$x = u\cdot v = \frac{\ln|p| + c}{\sqrt{p}} \quad (*)$$

$$y = -xp + 4\sqrt{p} = -p\frac{c + \ln|p|}{\sqrt{p}} + 4\sqrt{p}$$

$$y = \sqrt{p}(4 - c - \ln|p|) \quad (**)$$

(*) i (**) je rješenje dif. jedn. u parametarskom obliku

2. Riješiti diferencijalnu jednačinu $y'(2x - y) = y$.

Rj. $\left.\begin{aligned} x &= \frac{2}{3}p + \frac{c}{p^2} \\ y &= 2xp - p^2 \end{aligned}\right\}$ opšte rješ. dif. jedn. u parametarskom obliku

3. Naći rješenje diferencijalne jednačine $y = xy' - 2 - y'$ koje prolazi kroz tačku $A(2,5)$.

Rj. $y = xc - 2 - c$ opšte rješenje

$y = 7x - 9$ partikularno rješenje

Riješiti diferencijalnu jednačinu $2y+y'(2x+y')=0$.

Rj.

$y=-xy'-\frac{1}{2}(y')^2$ ovo je Lagranžova diferenc. jedn. $(y=xf(y')+g(y'))$

uvodimo smjene $y'=p$, $x=uv$

$y'=p$

$y=-xp-\frac{1}{2}p^2 \quad /\frac{d}{dx}$

$y'=-p-xp'-\frac{1}{2}\cdot 2pp'$

$p=-p-xp'-pp'$

$2p=(-x-p)p' \quad /:p'$

$\frac{2p}{p'}=-x-p$, $p'=\frac{dp}{dx}$

$\frac{1}{p'}=\frac{dx}{dp}=x'$

$x'=-\frac{1}{2p}x-\frac{1}{2}$

$x'+\frac{1}{2p}x=-\frac{1}{2}$ ovo je linearna dif. jedn. $(y'+f(x)y=g(x))$

uvodimo smjenu $x=uv$, $x'=u'v+uv'$

$u'v+uv'+\frac{1}{2p}uv=-\frac{1}{2}$

$u'v+u\underbrace{\left(v'+\frac{1}{2p}v\right)}_{=0}=-\frac{1}{2}$

$v'+\frac{1}{2p}v=0$

$\frac{dv}{dp}=-\frac{1}{2p}v$

$\frac{dv}{v}=-\frac{1}{2}\cdot\frac{dp}{p} \quad //\int$

$\ln|v|=-\frac{1}{2}\ln|p|$

$v=p^{-\frac{1}{2}}=\frac{1}{\sqrt{p}}$

$u'v=-\frac{1}{2}$

$u'\cdot\frac{1}{\sqrt{p}}=-\frac{1}{2}$

$\frac{du}{dp}=-\frac{1}{2}\sqrt{p}$

$du=-\frac{1}{2}p^{\frac{1}{2}}dp \quad //\int$

$u=-\frac{1}{2}\cdot\frac{p^{\frac{3}{2}}}{\frac{3}{2}}+c$

$u=-\frac{1}{2}\cdot\frac{2}{3}\sqrt{p^3}+c$

$u=-\frac{1}{3}p\sqrt{p}+c$

$x=uv=\left(-\frac{1}{3}p\sqrt{p}+c\right)\cdot\frac{1}{\sqrt{p}}$

$x=-\frac{p}{3}+\frac{c}{\sqrt{p}}$

$y=-xp-\frac{1}{2}p^2$

$y=\left(\frac{p}{3}-\frac{c}{\sqrt{p}}\right)p-\frac{1}{2}p^2$

$y=-\frac{1}{6}p^2-\frac{p}{\sqrt{p}}c$.

$$\left.\begin{aligned} x&=-\frac{p}{3}+\frac{c}{\sqrt{p}}\\ y&=-\frac{p^2}{6}-\frac{p}{\sqrt{p}}c\end{aligned}\right\}$$ opće rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu $y'+\frac{1}{y'}=\frac{y}{x}$.

Rj. $y'+\frac{1}{y'}=\frac{y}{x} \quad /\cdot x$

$y=xy'+\frac{x}{y'}$ uvodimo smjenu $y'=p$

$y=xy'+\frac{x}{y'}$

$y=x\left(y'+\frac{1}{y'}\right)$

ovo je Lagranžova diferencijalna jednačina

$y=xp+\frac{x}{p} \quad /\frac{d}{dx}$

$y'=p+xp'+\frac{p-xp'}{p^2}$ (kako je $y'=p$ imamo)

$p=p+xp'+\frac{1}{p^2}(p-xp')$

$xp'+\frac{1}{p}-\frac{xp'}{p^2}=0$

$\left(x-\frac{x}{p^2}\right)p'=-\frac{1}{p} \quad /\cdot p$

$\left(px-\frac{x}{p}\right)p'=-1 \quad /\cdot\frac{1}{p'}$

$-\frac{1}{p'}=px-\frac{1}{p}x$

$-\frac{1}{p'}=\left(p-\frac{1}{p}\right)x$

znamo da je $\frac{1}{p'}=\frac{1}{\frac{dp}{dx}}=\frac{dx}{dp}=x'$

pa imamo

$-x'=\left(p-\frac{1}{p}\right)x \quad /\cdot(-1)$

$x'=\left(\frac{1}{p}-p\right)x$ ovo je diferencijalna jednačina sa razdvojenim promjenjivim

$x'=\left(\frac{1}{p}-p\right)x$

$\frac{dx}{dp}=\left(\frac{1}{p}-p\right)x$

$\frac{dx}{x}=\left(\frac{1}{p}-p\right)dp \quad //\int$

$\int\frac{dx}{x}=\int\left(\frac{1}{p}-p\right)dp$.

$\ln|x|=\ln|p|-\frac{p^2}{2}+C_1$

$\ln|x|=\ln|p|+\ln e^{-\frac{p^2}{2}}+\ln C$

$x=pCe^{-\frac{p^2}{2}}$

$y=xp+\frac{x}{p}=Cpe^{-\frac{p^2}{2}}\cdot p+\frac{pCe^{-\frac{p^2}{2}}}{p}$

$y=Cp^2e^{-\frac{p^2}{2}}+C\cdot e^{-\frac{p^2}{2}}$

$y=Ce^{-\frac{p^2}{2}}(p^2+1)$

$$\left.\begin{aligned} x&=pCe^{-\frac{p^2}{2}}\\ y&=Ce^{-\frac{p^2}{2}}(p^2+1)\end{aligned}\right\}$$ opšte rješenje

Clairautova diferencijalna jednačina

je oblika $y = xy' + f(y')$

ove diferencijalne jednačine rješavamo na isti način kao što smo rješavali Lagranžovu dif. jedn.

uvodimo smjenu $y' = p$, $x = uv$
$dy = p\,dx$

(1.) Riješiti diferencijalnu jednačinu $xy' + \sin y' - y = 0$

Rj. $y = xy' + \sin y'$

$$y' = p \Rightarrow \frac{dy}{dx} = p \Rightarrow dy = p\,dx$$

$$y = xp + \sin p \quad / d$$

$$dy = p\,dx + x\,dp + \cos p\,dp$$

$$\underline{p\,dx} = \underline{p\,dx} + x\,dp + \cos p\,dp$$

$$x\,dp + \cos p\,dp = 0$$

$$dp(x + \cos p) = 0$$

$$dp = 0 \Rightarrow p = c$$

$y = cx + \sin c$ opšte rješenje dif. jedn.

(2.) Riješiti diferencijalnu jednačinu $y - xy' - \frac{y'^2}{2} = 0$.

Rj. $y = cx + \frac{c^2}{2}$ opšte rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $2y - 2xy' = a\left(\sqrt{1+(y')^2} - y'\right)$.

Rj. Lagranžova diferencijalna jednačina je oblika $y = xf(y') + g(y')$

$$2y - 2xy' = a\left(\sqrt{1+(y')^2} - y'\right)$$

$$2y = 2xy' + a\left(\sqrt{1+(y')^2} - y'\right) \quad /:2$$

$$y = xy' + \frac{a}{2}\left(\sqrt{1+(y')^2} - y'\right)$$

Ovo je Klerova diferencijalna jednačina

Uvodimo smjenu $y' = p$

$$y = xp + \frac{a}{2}\left(\sqrt{1+p^2} - p\right) \quad / \frac{d}{dx}$$

$$y' = p + xp' + \frac{a}{2}\left(\frac{\cancel{2}pp'}{\cancel{2}\sqrt{1+p^2}} - p'\right)$$

$y' = p$

$$p = p + xp' + \frac{a}{2}p'\left(\frac{p}{\sqrt{1+p^2}} - 1\right)$$

$$-xp' = \frac{a}{2}p'\left(\frac{p}{\sqrt{1+p^2}} - 1\right)$$

$$\left[x + \frac{a}{2}\left(\frac{p}{\sqrt{1+p^2}} - 1\right)\right]p' = 0$$

a) Ako je $p' = 0$ imamo da je $p = c$ tj. $y' = c$ pa iz $y = xy' + \frac{a}{2}\left(\sqrt{1+(y')^2} - y'\right)$

$$\Rightarrow y = xc + \frac{a}{2}\left(\sqrt{1+c^2} - c\right)$$

$y = xC_1 + \frac{a}{2}C_2$ opšte rješenje diferencijalne jednačine

b) Ako je $x + \frac{a}{2}\left(\frac{p}{\sqrt{1+p^2}} - 1\right) = 0$

$$\frac{p}{\sqrt{1+p^2}} - 1 = -\frac{2}{a}x$$

$$\frac{p}{\sqrt{1+p^2}} = 1 - \frac{2x}{a}$$

$$p^2 = \left(1 - \frac{2x}{a}\right)^2(1+p^2)$$

$$p^2 - \left(1 - \frac{2x}{a}\right)^2 p^2 = \left(1 - \frac{2x}{a}\right)^2$$

$$p^2 = \frac{\left(1 - \frac{2x}{a}\right)^2}{1 - \left(1 - \frac{2x}{a}\right)^2}$$

$$p = \frac{1 - \frac{2x}{a}}{\sqrt{1 - \left(1 - \frac{2x}{a}\right)^2}}$$

$$y = xy' + \frac{a}{2}\left(\sqrt{1+(y')^2} - y'\right) \Rightarrow$$

$$\Rightarrow y = \frac{x - \frac{2}{a}x^2}{\sqrt{1 - \left(1 - \frac{2}{a}x\right)^2}} + \frac{a}{2}\left(\sqrt{1 + \frac{\left(1 - \frac{2}{a}x\right)^2}{1 - \left(1 - \frac{2x}{a}\right)^2}} - \frac{1 - \frac{2x}{a}}{\sqrt{1 - \left(1 - \frac{2x}{a}\right)^2}}\right)$$

Kako se ovo rješenje ne može dobiti iz opšteg rješenja ovo je singularno rješenje

zadnji izraz se može pojednostaviti

$$Y=\frac{x-\frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}+\frac{a}{2}\left(\sqrt{\frac{1-\cancel{\left(1-\frac{2}{a}x\right)^2}+\cancel{\left(1-\frac{2}{a}x\right)^2}}{1-\left(1-\frac{2}{a}x\right)^2}}-\frac{1-\frac{2}{a}x}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}\right)$$

$$Y=\frac{x-\frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}+\frac{\frac{a}{2}}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}-\frac{\frac{a}{2}-x}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}.$$

$$Y=\frac{2x-\frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}$$ singularno rješenje dif. jedn.

Dio tablice integrala

1. $\int u^a\,du=\frac{u^{a+1}}{a+1}+C,\quad a\neq-1.$

2. $\int u^{-1}\,du=\int\frac{du}{u}=\int\frac{u'}{u}\,dx=$ $=\ln|u|+C.$

3. $\int a^u\,du=\frac{a^u}{\ln a}+C;\ \int e^u\,du=$ $=e^u+C.$

4. $\int \sin u\,du=-\cos u+C.$

5. $\int \cos u\,du=\sin u+C.$

6. $\int \sec^2 u\,du=\operatorname{tg} u+C.$

7. $\int \operatorname{cosec}^2 u\,du=-\operatorname{ctg} u+C.$

8. $\int\frac{du}{u^2+a^2}=\frac{1}{a}\operatorname{arc\,tg}\frac{u}{a}+C.$

9. $\int\frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$

10. $\int\frac{du}{\sqrt{a^2-u^2}}=\arcsin\frac{u}{a}+C.$

11. $\int\frac{du}{\sqrt{u^2+a}}=$ $=\ln\left|u+\sqrt{u^2+a}\right|+C.$

70 ispitnih zadataka za vježbu podjeljenih po oblastima - detaljno raspisana rješenja ovih zadataka možete skinuti sa stranice pf.unze.ba\nabokov\za_vjezbu

Sadržaj

1 Determinante

1. Izračunati determinantu $D = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 4 & 4 \\ -2 & -2 & -2 & 1 \\ 3 & 3 & 6 & x^2+3 \end{vmatrix}$, a zatim riješiti nejednačinu $D < 2x$.

2. Izračunati determinantu $D = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 6 & 5 \\ -1 & -1 & 0 & 2 \\ -3 & -3 & -6 & x^2-3 \end{vmatrix}$, a zatim riješiti nejednačinu $D < 2x$.

3. Izračunati determinantu $D = \begin{vmatrix} x^2-8 & 2 & 3 & 1 \\ 4 & 2 & 3 & 1 \\ -4 & -2 & 1 & 0 \\ 5 & 3 & 2 & 1 \end{vmatrix}$, a zatim riješiti nejednačinu $D < -x$.

4. Izračunati determinantu $D = \begin{vmatrix} 2 & 1 & 1 & -1 \\ 4 & x^2-1 & 1 & -1 \\ -4 & -2 & -1 & 0 \\ 5 & 3 & 2 & -1 \end{vmatrix}$, a zatim riješiti nejednačinu $D > x$.

5. Izračunati determinantu $D = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 3 & -7 & 6 & 2 \\ -4 & 6 & -9 & -6 \\ 5 & -7 & 12 & x^2 \end{vmatrix}$, a zatim riješiti nejednačinu $D > -2x$.

6. Izračunati determinantu $D = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 3 & -7 & 6 & 2 \\ -4 & 6 & x^2-13 & -6 \\ 5 & -7 & 12 & 7 \end{vmatrix}$, a zatim riješiti nejednačinu $D < 4x$.

2 Matrične jednačine

7. Riješiti matričnu jednačinu $BX = A + I$ ako je $A = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & -3 \\ 6 & 9 & -1 \end{bmatrix}$ i $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

8. Riješiti matričnu jednačinu $2I + BX = A$ ako je $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -3 & 1 \\ -3 & -6 & 4 \end{bmatrix}$ i $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

9. Riješiti matričnu jednačinu $-3X = 2AX + I$ ako je $A = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ i $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

10. Riješiti matričnu jednačinu $AX+I=-3X$ ako je $A=\begin{bmatrix}11 & -2 & -13\\ -6 & -2 & 5\\ -1 & 0 & -2\end{bmatrix}$ i $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

11. Riješiti matričnu jednačinu $I+AX=-2X$ ako je $A=\begin{bmatrix}-8 & -2 & 9\\ -3 & -3 & 4\\ 1 & 0 & -3\end{bmatrix}$ i $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

12. Riješiti matričnu jednačinu $-2X=3AX-I$ ako je $A=\begin{bmatrix}-1 & 0 & 0\\ 1 & -1 & 0\\ 4 & -1 & -1\end{bmatrix}$ i $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

13. Riješiti matrične jednačine

(a) $X\cdot(-2)=\begin{bmatrix}1&0\\0&1\end{bmatrix}+X\begin{bmatrix}1&2&3\\-4&-5&-6\end{bmatrix}\begin{bmatrix}-1&-8\\7&1\\-2&-4\end{bmatrix}$;

(b) $X\begin{bmatrix}-1&-2&-3\\4&5&6\end{bmatrix}\begin{bmatrix}1&8\\-7&-1\\2&4\end{bmatrix}=X\cdot(-1)+\begin{bmatrix}1&0\\0&1\end{bmatrix}$;

(c) $\begin{bmatrix}-1&2&-3\\4&-5&6\end{bmatrix}\begin{bmatrix}1&-8\\7&-1\\2&-4\end{bmatrix}X=\begin{bmatrix}1&0\\0&1\end{bmatrix}-2X$;

(d) $3X=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}-1&8\\-7&1\\2&4\end{bmatrix}X+\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

3 Sistemi linearnih jednačina

14. Rješiti sistem linearnih jednačina

$$\begin{aligned}x-y+z&=3\\x-y-z&=4\\x+y-z&=5\\x+y+z&=6\ .\end{aligned}$$

15. Rješiti sistem linearnih jednačina

$$\begin{aligned}x-y+z&=2\\x-y-z&=3\\x+y-z&=4\\x+y+z&=5\ .\end{aligned}$$

16. Rješiti sistem linearnih jednačina

$$\begin{aligned}x-y+z&=1\\x-y-z&=2\\x+y-z&=3\\x+y+z&=4\ .\end{aligned}$$

17. Riješiti sistem jednačina

$$\begin{aligned}x_1+2x_2-x_3-x_4-x_5&=10\\x_1+3x_2+x_3+x_4+x_5&=20\\-3x_1-6x_2+4x_3+4x_4+4x_5&=-27\ .\end{aligned}$$

18. Riješiti sistem jednačina

$$\begin{aligned}x_1-2x_2+x_3+x_4+x_5&=0\\-x_1+3x_2-3x_3-3x_4-3x_5&=-2\\3x_1-6x_2+4x_3+4x_4+4x_5&=3\ .\end{aligned}$$

19. Riješiti sistem jednačina

$$\begin{aligned}x_1+2x_2-4x_3-8x_4-12x_5&=-11\\-2x_1-3x_2+5x_3+10x_4+15x_5&=7\\-3x_1-5x_2+10x_3+20x_4+30x_5&=25.\end{aligned}$$

20. Riješiti sistem jednačina

$$\begin{aligned}x_1+2x_2-4x_3+8x_4+12x_5&=-10\\3x_1+7x_2-15x_3+30x_4+45x_5&=-43\\-2x_1-3x_2+6x_3-12x_4-18x_5&=13.\end{aligned}$$

21. Riješiti sistem jednačina

$$\begin{aligned}x_1+2x_2-4x_3-4x_4-16x_5&=-9\\2x_1+5x_2-11x_3-11x_4-44x_5&=-29\\-4x_1-7x_2+14x_3+14x_4+56x_5&=30.\end{aligned}$$

22. Riješiti sistem jednačina

$$\begin{aligned}x_1+2x_2-4x_3+16x_4-4x_5&=-8\\-2x_1-3x_2+5x_3-20x_4+5x_5&=7\\4x_1+9x_2-18x_3+72x_4-18x_5&=-37.\end{aligned}$$

4 Vektorski prostor

23. Dat je skup $\mathcal{B}=\left\{\begin{bmatrix}3\\-6\\-9\end{bmatrix},\begin{bmatrix}2\\-5\\-6\end{bmatrix},\begin{bmatrix}-1\\1\\5\end{bmatrix}\right\}$. Provjeriti da li je skup $\mathcal{B}$ linearno nezavisan.

Objasniti zašto je $\mathcal{B}$ baza vektorskog prostora $\mathbb{R}^3$? Vektor $u=\begin{bmatrix}1\\-1\\-1\end{bmatrix}$ izraziti kao linearnu kombinaciju vektora iz baze $\mathcal{B}$ (drugim riječima, odrediti koordinate vektora u u odnosu na bazu $\mathcal{B}$).

24. Dat je skup $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. Provjeriti da li je skup $\mathcal{B}$ linearno nezavisan. Objasniti zašto je $\mathcal{B}$ baza vektorskog prostora $\mathbb{R}^3$? Vektor $u = \begin{bmatrix} 3 \\ -4 \\ 4 \end{bmatrix}$ izraziti kao linearnu kombinaciju vektora iz baze $\mathcal{B}$ (drugim riječima, odrediti koordinate vektora u u odnosu na bazu $\mathcal{B}$).

25. Vektor $v \in \mathbb{R}$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \right\}$ ima koordinate $\begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$. Odrediti koordinate vektora v u odnosu na bazu $\mathcal{B}' = \left\{ \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

26. Vektor $v \in \mathbb{R}$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ ima koordinate $\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$. Odrediti koordinate vektora v u odnosu na bazu $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

27. Vektor $v \in \mathbb{R}$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ ima koordinate $\begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$. Odrediti koordinate vektora v u odnosu na bazu $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

28. Vektor $v \in \mathbb{R}$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ ima koordinate $\begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$. Odrediti koordinate vektora v u odnosu na bazu $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

29. Odrediti sve vrijednosti parametra m tako da vektori $\vec{a} = \begin{pmatrix} m-2 \\ 1 \\ 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} m-2 \\ m-2 \\ 3 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} m-2 & 1 & m-2 \end{pmatrix}^\top$, nisu baza (ne čine bazu) vektorskog prostora $\mathbb{R}^3$. Za najveću dobijenu vrijednost parametra m izraziti vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

30. Ako je $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ jedna baza vektorskog prostora $\mathbb{R}^3$, dokazati da i vektori $\vec{b}_1 = \vec{a}_2 + 3\vec{a}_3$, $\vec{b}_2 = \vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$ i $\vec{b}_3 = 2\vec{a}_1 + 2\vec{a}_2 + 6\vec{a}_3$ također čine bazu prostora $\mathbb{R}^3$ i izraziti vektor $\vec{c} = -\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$ preko vektora vaze $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$.

31. Odrediti sve vrijednosti parametra m tako da vektori $\vec{a} = \begin{pmatrix} m-1 \\ m-1 \\ m-1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ m-1 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 & 3 & m-1 \end{pmatrix}^\top$, nisu baza (ne čine bazu) vektorskog prostora $\mathbb{R}^3$. Za najveću dobijenu vrijednost parametra m izraziti vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

32. Ako je $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ jedna baza vektorskog prostora $\mathbb{R}^3$, dokazati da i vektori $\vec{b}_1 = \vec{a}_1 + 2\vec{a}_2 + 3\vec{a}_3$, $\vec{b}_2 = \vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$ i $\vec{b}_3 = 2\vec{a}_1 + \vec{a}_2 + 4\vec{a}_3$ također čine bazu prostora $\mathbb{R}^3$ i izraziti vektor $\vec{c} = \vec{a}_1 + \vec{a}_2 + \vec{a}_3$ preko vektora vaze $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$.

33. Za koje vrijednosti parametra m vektori $\vec{a} = (2m, 1+m, 1)^\top$, $\vec{b} = (-m, 1, m)^\top$, $\vec{c} = (m, 1, m-2)^\top$ čine bazu trodimenzionalnog vektorskog prostora?

34. Za koje vrijednosti parametra m vektori $\vec{a} = (m, -m, 1)^\top$, $\vec{b} = (-m, m, 2m+2)^\top$, $\vec{c} = (m, m+1, 1-m)^\top$ čine bazu trodimenzionalnog vektorskog prostora?

35. Za koje vrijednosti parametra m vektori $\vec{a} = (2m, 1-m, 1)^\top$, $\vec{b} = (-2m, m, 2m+2)^\top$, $\vec{c} = (m, 1+m, 1-m)^\top$ čine bazu trodimenzionalnog vektorskog prostora?

5 Limesi

36. Bez upotrebe H'Lopitalovog pravila izračunati limese

$$(a) \lim_{x\to 3} \frac{3x^2-10x+3}{2x^2-7x+3}; \quad (b) \lim_{x\to -\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin^3 x}; \quad (c) \lim_{x\to 7} \frac{2x^2-13x-7}{-2x^2+11x+21};$$

$$(d) \lim_{x\to \frac{\pi}{2}} \frac{1-\sin^3 x}{\cos^2 x}; \quad (e) \lim_{x\to 1} \frac{5x^2-3x-2}{7x^2-10x+3}; \quad (f) \lim_{x\to 0} \frac{1-\cos^3 x}{\sin^2 x};$$

$$(g) \lim_{x\to 5} \frac{2x^2-11x+5}{3x^2-14x-5}; \quad (h) \lim_{x\to \pi} \frac{\sin^2 x}{1+\cos^3 x}.$$

37. Bez upotrebe H'Lopitalovog pravila izračunati limese

$$(a) \lim_{x\to -2} \frac{-5x^2-30x-40}{-3x^2+6x+24}; \quad (b) \lim_{x\to -4} \frac{-4x^2-12x+16}{-2x^2-10x-8};$$

$$(c) \lim_{x\to -6} \frac{3x^2+12x-36}{2x^2+10x-12}; \quad (d) \lim_{x\to -8} \frac{5x^2+35x-40}{-2x^2-6x+80}.$$

6 Izvodi

38. Odrediti prvi izvod funkcije

$$(a)\ y = \ln\frac{x^2-1}{x+1} + \operatorname{arc\,tg} x^2 \quad (b)\ y = \ln\frac{x}{x-1} + \arcsin x^2 \quad (c)\ y = \ln\frac{x^2}{x+1} + \operatorname{tg} x^2$$

7 Ispitivanje funkcija

39. Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti funkcija

$$(a)\ y = \frac{(x-3)^3}{(x-4)^2} \quad (b)\ y = \frac{(x-2)^3}{(x+1)^2}$$

40. Odrediti kosu asimptotu sljedećih funkcija

$$(a)\ y = \frac{3x^4-x}{x^3+2}; \quad (b)\ y = \frac{x^4+1}{x^3-1}; \quad (c)\ y = \frac{2x^2-3x+4}{x-2}: \quad (d)\ y = \frac{2x^3+4}{x^2-x+1}.$$

41. Ispitati i nacrtati grafik sljedećih funkcija

$$(a)\ y = \frac{x-2}{x^2-8x+16}; \quad (b)\ y = \frac{x-5}{x^2-2x+1};$$

$(c)\ y=\dfrac{x-3}{x^2-4x+4};\qquad (d)\ y=\dfrac{x-1}{x^2-10x+25};$

42. Ispitati i grafički predstaviti sljedeće funkcije

$(a)\ y=\dfrac{x^3-2}{2x^2}$ (ima greška u rješenju ovog zadataka - prvi integral nije dobar);

$(b)\ y=\dfrac{x^2+10}{x^2+4x+4}.$

43. Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti funkcije

$$(a)\ y=\frac{\ln x}{x};\quad (b)\ y=\frac{1+\ln x}{x^2};\quad (c)\ y=\frac{1-\ln x}{x^2};\quad (d)\ y=\frac{1+\ln x}{\ln x};$$
$$(e)\ y=\frac{2+\ln x}{6x^2};\quad (f)\ y=\frac{3+\ln x}{x}.$$

44. Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti funkcije

$$(a)\ y=x^2e^{-\frac{x}{3}};\quad (b)\ y=xe^{-\frac{1}{x}};\quad (c)\ y=xe^{-\frac{x^2}{4}};\quad (d)\ y=xe^{-\frac{x}{2}}.$$

45. Odrediti ekstreme, prevojne tačke te intervale konveksnosti i konkavnosti funkcije

$$(a)\ y=\frac{e^{2x}}{x+1};\quad (b)\ y=\frac{e^{3x}}{1+e^{-x}};\quad (c)\ y=\frac{e^{2x}}{1+e^{2x}};\quad (d)\ y=\frac{e^{3x}}{1-x}.$$

46. Odrediti parametre a i b tako da je prava

$(a)\ y=x-4$ kosa asimptota funkcije $y=\dfrac{(ax+b)^4}{x^3};$

$(b)\ y=27x+9$ kosa asimptota funkcije $y=\dfrac{(ax+b)^3}{x^2};$

$(c)\ y=4x+4$ kosa asimptota funkcije $y=\dfrac{(ax+b)^2}{x};$

$(d)\ y=64x-27$ kosa asimptota funkcije $y=\dfrac{a^2x^3+b^3x^2+1}{x^2}.$

47. Odrediti definiciono područje, znak te ekstreme funkcije

$$(a)\ y=\ln\frac{x}{x^2-1};\quad (b)\ y=\ln\frac{x-1}{x^2+1};\quad (c)\ y=\ln\frac{x^2-1}{x+1};\quad (d)\ y=\ln\frac{x+1}{x-1}.$$

48. Ispitati i grafički predstaviti sljedeće funkcije

$(a)\ y=\dfrac{\ln^2 x+1}{x^2};$

$(b)\ y=\ln\dfrac{x^2-3x+2}{x^2+1}.$

8 Ekstremi funkcija dvije promjenjive

49. Odrediti stacionarne tačke funkcije

$$(a)\ z=\frac{1}{2}x^2-xy+xy^2-\frac{1}{2}x^2y;\qquad (b)\ z=9x^2-\frac{9}{2}x^2y+6xy^2-12xy;$$

$$(c)\ z=x^2y-\frac{1}{2}xy^2-xy+\frac{1}{2}y^2;\qquad (d)\ z=6x^2y-\frac{9}{2}xy^2-12xy+9y^2.$$

50. Naći ekstreme funkcije $z=x^3+3xy^2-15x-12y$.

51. Odrediti ekstreme funkcije

$$(a)\ z=x^2+y^3+4x\sqrt{x^3}-3y;\qquad (b)\ z=3\ln\frac{x}{6}+\ln(12-y-x)+2\ln y;$$
$$(c)\ z=x^3+y^2-3x+4\sqrt{y^5};\qquad (d)\ z=2\ln x+\ln(12-x-y)+3\ln\frac{y}{6}.$$

9 Neodređeni integrali

52. Odrediti integrale $(a)\ I=\displaystyle\int\frac{\sin x\cdot\cos x}{e^x}\mathrm{d}x,\ (b)\ \int\frac{\sin x+\cos x}{\sin x+2\cos x}\mathrm{d}x.$

53. Odrediti integrale

$$(a)\ \int(x^2+2x)\cos 2x\,\mathrm{d}x,\qquad (b)\ \int(\frac{3}{2}x^2+3x)\sin 3x\,\mathrm{d}x,$$
$$(c)\ \int x\,\mathrm{arc\,tg}\,x\,\mathrm{d}x,\qquad (d)\ \int x\,\mathrm{arc\,ctg}\,x\mathrm{d}x.$$

54. Odrediti integrale

$$(a)\ \int\frac{(5x-3)\,dx}{\sqrt{-34+12x-x^2}},\qquad (b)\ \int\frac{(4x+2)\,dx}{\sqrt{-22+10x-x^2}},$$
$$(c)\ \int\frac{(2x-1)\,dx}{\sqrt{-7+6x-x^2}},\qquad (d)\ \int\frac{(3x-7)\,dx}{\sqrt{-33+12x-x^2}}.$$

55. Odrediti integrale

$$(a)\ \int\frac{\mathrm{d}x}{3x-4\sqrt{x}},\qquad (b)\ \int\frac{\sqrt{x}\,\mathrm{d}x}{\sqrt[4]{x^3}+4},$$
$$(c)\ \int\frac{\sqrt{x}\,dx}{\sqrt[3]{x^2}+\sqrt[4]{x}},\qquad (d)\ \int\frac{\sqrt[6]{x+1}\,\mathrm{d}x}{\sqrt{x+1}+\sqrt[3]{x+1}}.$$

56. Odrediti integrale

$$(a)\ \int x\ln(x-1)\mathrm{d}x,\qquad (b)\ \int\ln(1+x^2)\mathrm{d}x,$$
$$(c)\ \int\ln(x^2-1)\mathrm{d}x,\qquad (d)\ \int(x+1)\ln x\mathrm{d}x.$$

57. Odrediti integral

$$(a)\ \int\frac{7x-17}{x^2-5x+6}\mathrm{d}x\qquad (b)\ \int\frac{9x-2}{x^2-x-6}\mathrm{d}x\qquad (c)\ \int\frac{11x+14}{x^2+3x-4}\mathrm{d}x$$

10 Određeni integral

58. Izračunati integrale

(a) $\int\limits_{-\pi/2}^{\pi/2} x|\cos x|\,\mathrm{d}x$, (b) $\int\limits_{-\pi/2}^{\pi/2} e^x|\cos x|\,\mathrm{d}x$,

(c) $\int\limits_{0}^{\pi} x|\sin x|\,\mathrm{d}x$, (d) $\int\limits_{0}^{\pi} e^x|\sin x|\,\mathrm{d}x$.

11 Primjena određenog integrala

59. Primjenom određenog integrala odrediti površinu figure koju ograničava x-osa zajedno sa linijama $x + 3y - 3 = 0$, $x = -3$ i $x = 6$.

60. Primjenom određenog integrala odrediti površinu figure koju ograničava x-osa zajedno sa linijama $-x - 2y + 2 = 0$, $x = -4$ i $x = 2$.

61. Primjenom određenog integrala odrediti površinu figure koju ograničava y-osa zajedno sa linijama $x + y - 1 = 0$, $y = 3$ i $y = -2$.

62. Izračunati površinu ravne figure koja je ograničena linijama $y = -x^2$ i $x - y - 2 = 0$.

63. Izračunati površinu ravne figure koja je ograničena parabolama

(a) $y = 4 - x^2$ i $y = x^2 - 2x$; (b) $y = -x^2 - 4x$ i $y = x^2 + 2x$;

(c) $x = y^2 - 1$ i $x = -y^2 - 2y + 3$; (d) $x = y^2 - 4y + 3$ i $x = -y^2 + 2y + 3$.

64. Odrediti površinu figure ograničene

(a) hiperbolom $xy = 4$ i pravom $y = -x + 5$.

(b) parabolom $y = x^2 + 4x$ i pravom $x - y + 4 = 0$.

(c) parabolom $4y = 8x - x^2$ i pravom $4y = x + 6$.

(d) hiperbolom $xy = 6$ i pravom $y = 7 - x$.

12 Diferencijalne jednačine

65. Odrediti opšte rješenje sljedećih diferencijalnih jednačina

(a) $x^2(y+1)\mathrm{d}x + y^2(x-1)\mathrm{d}y = 0$; (b) $4x\mathrm{d}y - y\mathrm{d}x = x^2\mathrm{d}y$;

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{4y}{x(y-3)}$.

66. Odrediti partikularno rješenje diferencijalne jednačine $(1 + x^3)\mathrm{d}y - x^2y\mathrm{d}x = 0$ koje zadovoljava inicijalni uslov $x = 1$, $y = 2$.

67. Odrediti opšte rješenje date diferencijalne jednačine

(a) $y' = \dfrac{x+y}{x-y}$; (b) $y' = \dfrac{y^2}{x^2} - 2$;

(c) $x\,dy - y\,dx = y\,dy$; (d) $y' = \dfrac{2xy}{x^2 - y^2}$.

68. Odrediti opšte rješenje diferencijalne jednačine

(a) $xy' - \dfrac{y}{x+1} = x$ koje zadovoljava uslov $y(1) = 0$;

(b) $y' - y\,tg\,x = \dfrac{1}{cos\,x}$ koje zadovoljava uslov $y(0) = 0$;

(c) $xy' + y - e^x = 0$ koje zadovoljava uslov $y(a) = b$;

(d) $xy' - 3y = x^4e^x$ koje zadovoljava uslov $y(1) = e$.

69. Odrediti opšte rješenje diferencijalne jednačine $2x^3y' = 2x^2y - y^3$.

70. Odrediti opšte rješenje datih diferencijalnih jednačina

(a) $y - xy' - \dfrac{1}{2}y'^2 = 0$;

(b) $y'^2 - xy' + y = 0$;

(c) $(y - y'x)^2 = 1 + y'^2$;

(d) $y = y'x + \sqrt{4 + y'^2}$.